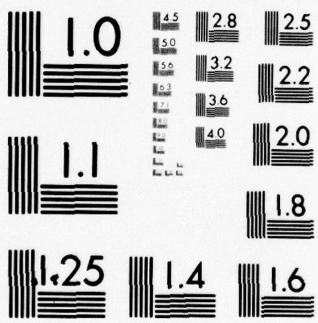
AD-A077 693 DAYTON UNIV OH RESEARCH INST F/G 20/4
FAR FIELD CONDITIONS FOR SUBSONIC FLOWS WITH SMALL SUPERIMPOSED--ETC(U)
JUL 79 K G GUDERLEY AFOSR-78-3524
UNCLASSIFIED UDR-TR-79-65 AFFDL-TR-79-3109 NL

| OF |
ADA 077693

END
DATE FILMED
1 -80
DDC

END
DATE FILMED
1 -80
DDC



MICROCOPY RESOLUTION TEST CHART

AFFDL-TR-79-3109

FAR FIELD CONDITIONS FOR SUBSONIC FLOWS WITH SMALL SUPERIMPOSED HARMONIC OSCILLATIONS

KARL G. GUDERLEY
UNIVERSITY OF DAYTON RESEARCH INSTITUTE
300 COLLEGE PARK AVENUE
DAYTON, OHIO 45469

JULY 1979

TECHNICAL REPORT AFFDL-TR-79-3109
Interim Report for period December 1978 to April 1979

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

CHARLES L. KELLER
Project Engineer
Applied Mathematics Group
Analysis and Optimization Branch

DALE E. COOLEY, Atcg Branch
Chief, Analysis and Optimization Br
Structures and Dynamics Division

FOR THE COMMANDER

RALPH L. KUSTER, Jr, Col, USAF
Chief, Structures and Dynamics Division

"If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify AFFDL/FBRD *,W-PAFB, OH 45433 to help us maintain a current mailing list".*

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 **REPORT DOCUMENTATION PAGE**

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFFDL-TR-79-3109	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER: 9
4. TITLE (and Subtitle): 6 Far Field Conditions for Subsonic Flows with Small Superimposed Harmonic Oscillations.	5. TYPE OF REPORT & PERIOD COVERED: Interim Report. Dec [redacted] 78 — Apr [redacted] 79,
	6. PERFORMING ORG. REPORT NUMBER: 14 UDR-TR-79-65
7. AUTHOR(s): 10 Karl G. Guderley	8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-78-3524
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Dayton Research Institute, 300 College Park Avenue, Dayton, Ohio 45469	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304 N102 17 N1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Flight Dynamics Laboratory/FBRD, Wright-Patterson Air Force Base, Ohio 45433	12. REPORT DATE: 11 Jul [redacted] 979
	13. NUMBER OF PAGES: 64 12 64
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Transonic Flows
Oscillating Flows
Far Field Conditions
Helmholtz Equation
Radiation Condition

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For numerical work one needs boundary conditions (so called far field conditions) at the outer edge (or surface) of the region of the flow field for which the computations are carried out. In oscillatory flows with higher frequencies and Mach numbers close to 1 these conditions become crucial and the formulation in use at present which is patterned after far field conditions

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

105400

JOB

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20. ABSTRACT (continued)

in steady flows becomes impractical if not useless. The far field conditions express the fact that no waves coming from infinity are present. Mathematically this appears in the form of the requirement that the far field can be represented by a superposition of particular solutions which represent outgoing waves. On this basis one obtains by the application of Green's formula a number of integral conditions. An example shows that even with this formulation one will encounter numerical difficulties (because of latent standing waves) if one tries to solve the flow equations by iteration. Fortunately the new far field conditions are well suited to a solution of the problem by direct elimination, possibly combined with a refinement of the results by the multigrid method of Brand. This report clarifies the basic concepts and examines the mathematical background. This includes the derivation of particular solutions necessary to express the far field conditions, the relations between these particular solutions and outgoing acoustical waves, and the relation between the derivatives of certain of these particular solutions and particular solutions of a higher order. Most of the analysis is carried out for the three dimensional problem. This requires a rather thorough discussion of Legendre polynomials. The conditions for the two-dimensional problem are obtained by rather obvious specialization.

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

FOREWORD

The work was performed under Grant AFOSR-78-3524 to the University of Dayton with K. G. Guderley as Principal Investigator in 1978-1979.

The author expresses his appreciation for the excellent typing work of Miss Norma Harting and for the art work of the staff of the University of Dayton.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I
INTRODUCTION

By the very nature of a numerical approach the computation of a flow field is restricted to a finite region, Figure 1. At the outer boundary of this region one must impose conditions which insure that a further continuation of the flow field has the properties desired at infinity. In a steady flow this means that the deviations from a parallel flow tend to zero as one goes to infinity. In an oscillatory flow this criterion is not sufficient, for it is satisfied by incoming as well as by outgoing waves. One must express, somehow, that only outgoing waves are permissible.

The present discussions are carried out with a view toward the computation of flow fields in the lower transonic region. The flow in the far field does not show the difficulties which are typical for the transonic regime; it can be treated by means of the potential equation for unsteady flows linearized for the vicinity of a parallel flow with the prescribed free stream Mach number. In steady flow, one deals in essence, with the Laplace equation. If oscillations with a fixed frequency are present, one ultimately has to solve the Helmholtz equation.

The far field conditions for steady transonic flows have been studied by Klunker (Ref. 1). The extension to oscillatory flows is due to Traci, Albano, Farr and Cheng (Ref. 2). In one regard these formulations have an unexpected form. The conditions at the far boundary express a property of the far field. It should, therefore, be possible to formulate them in terms of quantities pertaining to the far field, specifically in terms of the potential and its normal derivative at its inner boundary. These quantities are, of course, identical with the potential and the normal derivative at the outer boundary of the computed part of the flow field. The formulations of Ref. 1 and Ref. 2 express the far field conditions in terms of near field data. A formulation solely in terms of far field data is indeed possible. It is not more complicated than that of Ref. 1 and 2. We shall give a rather thorough discussion of a number of different formulations including that of Klunker and Traci.

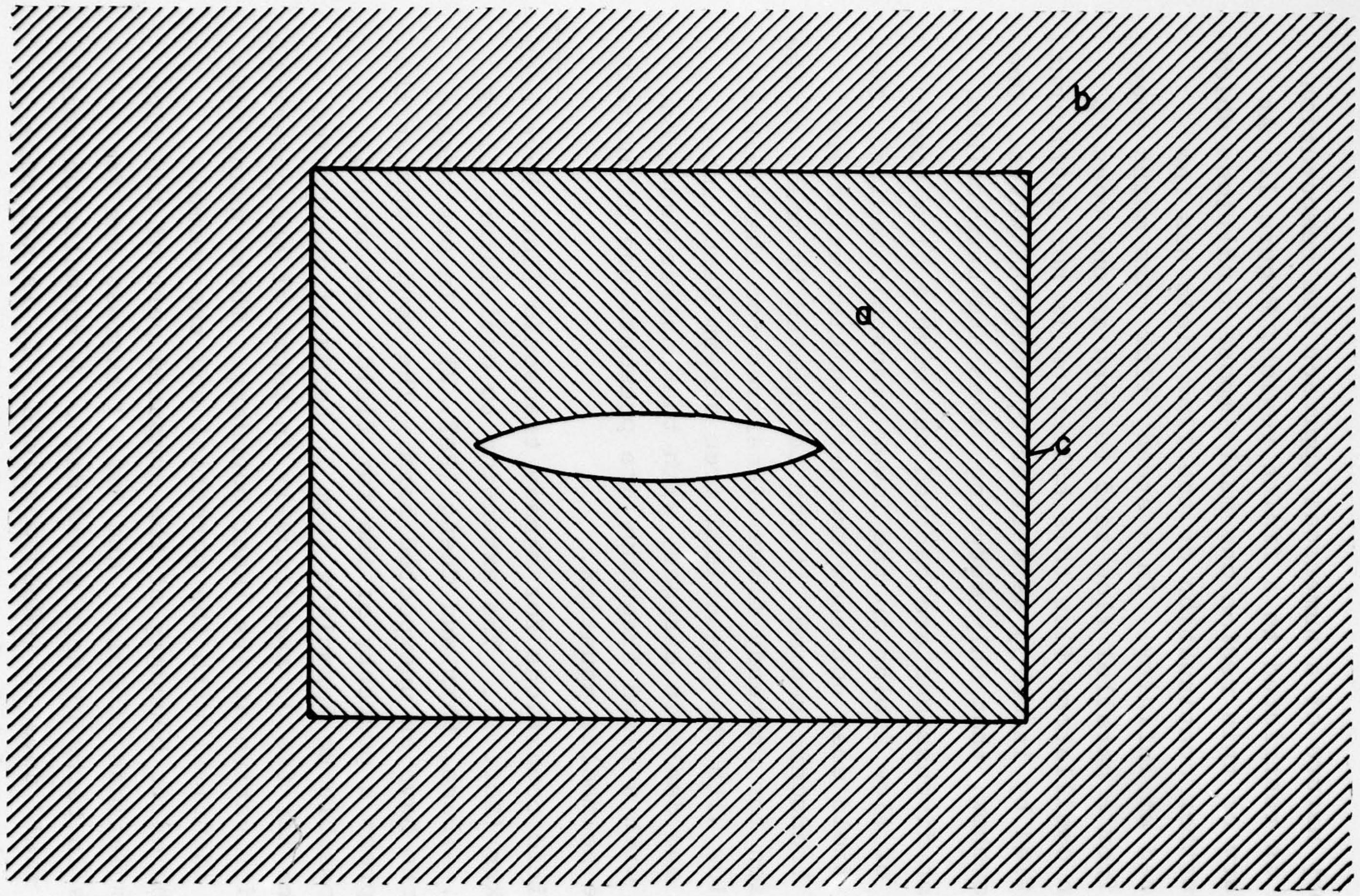


Figure 1. Near and Far Field.

a. Region for Which Computations are Carried Out (near field)

b. Far Field

c. Boundary Between the Near and the Far Field at Which Far Field Conditions are Applied.

A clear understanding of the far field conditions is particularly important for oscillatory flows. In a steady flow the conditions at the far boundary have only a relatively small influence. They can be taken into account by iteration. For this purpose the procedures of Ref. 1 and 2 are well suited. Actually they have been used only in this sense. If one has a flow with harmonic oscillations, then the far field condition has a drastic effect unless the frequency is very low. Take for instance the field around a pulsating sphere in air at rest (Helmholtz equation). The solution which admits only outgoing waves is quite different from the one where, at a large radius, the amplitude of the oscillation is required to be zero. For such flows, it is likely that one will have convergence difficulties and under these circumstances alternative formulations of the far field conditions may be useful.

The formulation of far field conditions obviously requires that the oscillatory part of the solution in the far field can be represented analytically. This is possible, only if the governing equation is rather simple. The Helmholtz equation, for instance, has simple fundamental solutions for the far field. The differential equations for oscillations superimposed to a flow with a free stream Mach number one and linearized for the vicinity of such a flow are much more complicated and analytic solutions are not available. Such problems are not included in the present analysis.

The primary goal of this report is the development of the basic concepts. The work is, of course, done with a view toward numerical implementation; however, it is still several steps removed from an actual program. An example is given, but it is overly simple. Its sole purpose is to illustrate the typical steps which one has to carry out. But it also shows some specific difficulties encountered in solving the Helmholtz equation. This is desirable because the experience gained with the Laplace equation is not always a safe intuitive guide.

Section II starts with the linearized equation for unsteady flow, carries out transformations which bring the problem into the form of the Helmholtz equation and derives particular solutions which presumably represent outgoing (or incoming) waves. Section III has a more theoretical nature. It shows that the particular solutions derived in Section II are a superposition of outgoing (or incoming) acoustical waves; that is, it shows that these particular solutions have indeed the desired characters. Section IV uses Green's formula to discuss certain properties of the solutions at the far boundary. The far field conditions are fully formulated in Section V. This includes a discussion of the formulations of Ref. 1 and 2. In Section VI the principal steps needed in an application are discussed on the basis of an example. It is very simple so that the steps which usually are carried out numerically can be replaced by analytical formulae. In these examples, one recognizes certain difficulties which may arise in practical work. Sections VII and VIII contain a number of observations concerning the practical side of the numerical work.

SECTION II

BASIC EQUATIONS AND PARTICULAR SOLUTIONS

The linearized equation for unsteady compressible flow is given by

$$\bar{\phi}_{\bar{x}\bar{x}}(1-U^2/a^2) + \bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} - 2(U/a^2)\bar{\phi}_{\bar{x}t} - (1/a^2)\bar{\phi}_{tt} = 0 \qquad (1)$$

$\bar{x}$, $\bar{y}$, $\bar{z}$ is a system of Cartesian coordinates, t the time, $\bar{\phi}$ the velocity potential, U the freestream velocity and "a" the velocity of sound (constant because of the linearization). We shall write

$$M = U/a \qquad (2)$$

One sets

$$\bar{\phi}(\bar{x},\bar{y},\bar{z},t) = \tilde{\phi}(\tilde{x},\tilde{y},\tilde{z}) \exp(i\nu t) \qquad (3)$$

where ν is a circular frequency, and $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ arise by the Prandtl Glauert coordinate distortion

$$\bar{x} = \tilde{x}; \quad \bar{y} = (1-M^2)^{-1/2}\tilde{y}, \quad \bar{z} = (1-M^2)^{-1/2}\tilde{z} \qquad (4)$$

This gives

$$\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} + \tilde{\phi}_{\tilde{z}\tilde{z}} - \frac{2i\nu U}{a^2-U^2}\tilde{\phi}_{\tilde{x}} + \frac{\nu^2}{a^2-U^2}\tilde{\phi} = 0 \qquad (5)$$

The hypothesis

$$\tilde{\phi} = \phi \exp\left(\frac{i\nu U}{a^2-U^2}\tilde{x}\right) \qquad (6)$$

removes the term $\tilde{\phi}_x$. The coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are made dimensionless with some characteristic length L (either the chord or the half chord of an airfoil).

$$x = \tilde{x}/L; \quad y = \tilde{y}/L; \quad z = \tilde{z}/L$$

Then one obtains the Helmholtz equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + \mu^2\phi = 0 \qquad (7)$$

with

$$\mu = \frac{a\nu L}{a^2-U^2} \qquad (8)$$

Frequently, one introduces in such problems a reduced frequency to be noted here by μ'

$$\mu' = \nu L/U$$

one has

$$\mu = \mu' \frac{M}{1-M^2} \tag{9}$$

Let $\vec{r}$ and $\vec{r}\,'$ be position vectors in the x, y, z system and let

$$r = |\vec{r} - \vec{r}\,'|$$

A fundamental solution to be denoted by $\omega(\vec{r}, \vec{r}\,')$ is obtained by the

$$\omega(\vec{r}, \vec{r}\,') = f(r)$$

and one obtains

$$\omega(\vec{r}, \vec{r}\,') = -\frac{4\Sigma}{r} \exp(-i\mu r) \tag{10}$$

or

$$\omega(\vec{r}, \vec{r}\,') = -\frac{4\Sigma}{r} \exp(+i\mu r) \tag{11}$$

The expressions 10 and 11 represent outgoing and incoming waves, respectively. At least they would represent outgoing and incoming waves, if the Helmholtz equation had been obtained by making the assumption

$$\varphi(x,y,z,t) = \omega(r) \exp(i\mu t)$$

in the equation for acoustical waves

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_{tt} = 0$$

The author did not find this analogy completely convincing, but we shall show in the next section that Eq. (10) can, indeed, be interpreted as a superposition of outgoing acoustical waves. For someone willing to accept this interpretation, there is no need for the discussion of the next section.

Next, we derive solutions corresponding to a family of particular solutions which include the counterparts of Eqs. (10) and (11) for the two dimensional case. We omit the z dependence in Eq. (7) and introduce polar coordinates

$$r = [(x-x')^2 + (y-y')^2]^{1/2}$$

$$\theta = \operatorname{arctg}[(y-y')/(x-x')] \tag{12}$$

(In most cases, one chooses x' = y' = 0.) One then obtains

$$\phi_{rr} + r^{-1}\phi_r + r^{-2}\phi_{\theta\theta} + \mu^2\phi = 0 \tag{13}$$

The hypothesis

$$\phi_m = g_m(r) \begin{cases} \cos(m\theta) \\ \sin(m\theta) \end{cases} \tag{14}$$

leads to

$$g_m'' + r^{-1} g_m' + (\mu^2 - m^2 r^{-2}) g_m = 0 \tag{15}$$

This is Bessel's equation. One then obtains the solutions

$$\phi_m = H_m^{(1)}(\mu r) \begin{cases} \cos(m\theta) \\ \sin(m\theta) \end{cases} \tag{16}$$

$$\phi_m = H_m^{(2)}(\mu r) \begin{cases} \cos(m\theta) \\ \sin(m\theta) \end{cases} \tag{17}$$

and, in particular, for m = 0

$$\omega(r, r') = \frac{-i}{4} H_0^{(2)}(\mu r) \tag{18}$$

$$\omega(r, r') = \frac{i}{4} H_0^{(1)}(\mu r) \tag{19}$$

The expressions H behave asympototically as

$$H_m^{(2)}(z) \sim \text{const}\, z^{-1/2} \exp(-iz) \tag{20}$$

$$H_m^{(1)}(z) \sim \text{const}\, z^{-1/2} \exp(+iz) \tag{21}$$

This suggests that the expressions 18 and 20 represent outgoing and the expressions 19 and 21, incoming waves.

Eq. (15) shows that g is oscillatory for $r > m/\mu$ and nonoscillatory for $r < m/\mu$. (In the nonoscillatory region, g is monotonic except for possibly one minimum of the absolute value of g.) If the nonoscillatory region of the solution resembles that of the Laplace equation, the region increases with m and decreases with increasing μ.

We derive the corresponding particular solutions also for the three dimensional case. Usually one needs them only for r'=0. We introduce a system of spherical coordinates with radius r, ψ longitude, and θ latitude and set ζ = sin θ, where the z axis points in the north direction; that is, we introduce

$$\begin{aligned} r^2 &= x^2+y^2+z^2 \\ \zeta &= z/r \\ \psi &= \operatorname{arctg}(y/x) \end{aligned} \tag{22}$$

then one obtains

$$\phi_{xx}+\phi_{yy}+\phi_{zz}+\mu^2\phi = \phi_{rr}+\frac{2}{r}\phi_r+\frac{1}{r^2}\frac{\partial}{\partial\zeta}[(1-\zeta^2)\phi_\zeta]+\frac{1}{r^2(1-\zeta^2)}\phi_{\psi\psi} + \mu^2\phi = 0 \tag{23}$$

The hypothesis

$$\phi(r,\zeta,\psi) = \phi_n^m(r,\zeta)\exp(im\psi) \tag{24}$$

gives

$$\frac{\partial^2}{\partial r^2}\phi_n^m+\frac{2}{r}\frac{\partial}{\partial r}\phi_n^m+\frac{1}{r^2}\frac{\partial}{\partial\zeta}[(1-\zeta^2)\frac{\partial}{\partial\zeta}\phi_n^m]+(\mu^2-\frac{m^2}{r^2(1-\zeta^2)})\phi_n^m = 0 \tag{25}$$

Next, an expression

$$\phi_n^m = f^{(n)}(r)\,g_n^m(\zeta) \tag{26}$$

is substituted into Eq. (25). Multiplying the result by $r^2/(f(r)g(\zeta))$ and introducing a separation constant λ, one then obtains

$$\frac{d^2f^{(n)}}{dr^2}+\frac{2}{r}\frac{df^{(n)}}{dr}+(\mu^2-\frac{\lambda}{r^2})f^{(n)} = 0 \tag{27}$$

and

$$\frac{d}{d\zeta}[(1-\zeta^2)\frac{dg_n^m}{d\zeta}]+(\lambda-\frac{m^2}{1-\zeta^2})g_n^m(\zeta) = 0 \tag{27'}$$

Eq. (27') is the equation of associated Legendre functions (spherical harmonics) usually denoted by P^m_n. Notice that

Eq. (27') does not contain the frequency μ. The separation constant is determined from Eq. (27') by the requirement that the function ϕ finally obtained be regular at the poles $\zeta = \pm 1$. One obtains

$$\lambda = n(n + 1)$$

(This explains the introduction of n anticipated in Eq. (26).) Moreover, one has

$$m \leq n$$

(see for instance Reference 4, Chapter 11). Substitution of λ into Equation 27 gives

$$\frac{d^2 f^{(n)}}{dr^2} + \frac{2}{r}\frac{df^{(n)}}{dr} + \left(\mu^2 - \frac{n(n+1)}{r^2}\right) f^{(n)} = 0 \tag{28}$$

This differential equation has a regular singular point with exponential n and -(n+1) at r = 0 and an essential singularity at infinity. The series development proceeds in powers of r^2. The function $f^{(n)}$ can be expressed in terms of elementary transcendental functions. One sets for this purpose

$$f^{(n)}(r) = r^{-(n+1)} \exp(-i\mu r) \hat{f}(r) \tag{29}$$

then one obtains

$$\frac{d^2\hat{f}}{dr^2} - \frac{2n}{r}\frac{d\hat{f}}{dr} - 2i\mu\left(\frac{d\hat{f}}{dr} - \frac{n}{r}\hat{f}\right) = 0$$

A power series hypothesis for $\hat{f}$

$$\hat{f} = \sum a_k r^k \tag{30}$$

leads to the recurrence relation

$$\frac{a_{k+1}}{a_k} = \frac{2i\mu(n-k)}{(k+1)(2n-k)} \tag{31}$$

hence

$$a_k = (2i\mu)^k \frac{n!}{(n-k)!} \frac{(2n-k)!}{(2n)!} \frac{1}{k!} \tag{32}$$

The coefficients are zero for $k < 0$, and for $k > n$ for the factorials of negative numbers are infinite. In other words, the function $\hat{f}$ is a polynomial. A second solution would start with a $2n + 1$. It is then given by an infinite series. But a second solution can be obtained in a different way (as we shall see).

For a steady flow ($\mu = 0$), $f^n(r)$ is given by r^{-n-1} and r^n. In the far field only the negative powers play a role. In a steady flow it is, therefore, possible to formulate far field conditions (where r is large) in terms of a very limited number of such particular solutions, usually only r^{-1} and r^{-2} (poles and dipoles). The contributions of single poles vanish if one has closed bodies. In oscillatory flows, all particular solutions decrease for large r and r^{-1}, it may therefore be necessary to take a large number of these particular solutions into account.

Eq. (28) shows that the solutions f^n are nonoscillatory for $(\mu r)^2 < n(n+1)$.

The differential equation for f^n (Eq. (28)) has real coefficients. It is therefore satisfied separately by the real and the imaginary parts of the expression Eq. (29).

If $a_0 = 1$ is real, it then follows from Eq. (30) that a_1 is imaginary. This means that the development of the imaginary part of the expression (29) has as lowest power r^{-n}. However, the exponents of the regular singular point $r = 0$ are $-n-1$ and $+n$. It follows that the lowest power in the development of the imaginary part is actually r^{+n}. The expression (29) contains, accordingly, the solution pertaining to the exponent $+n$ as well as that to the exponent $-n-1$.

For $n = 0$ one obtains the fundamental solution, Eq. (10). Because of the presence of the exponential function in Eq. (29), one will surmise that also these expressions are related to outgoing waves. (Of course the real or imaginary parts by themselves are a mixture of incoming and outgoing waves.)

Incoming waves are obtained by taking the conjugate complex of the expression 29. Let us introduce the notation

$$\left.\begin{aligned} \phi^{(m,n,1,1)} &= P_n^m(\xi)\cos(m\psi)\, f^{(n)}(r,\mu) \\ \phi^{(m,n,2,1)} &= P_n^m(\xi)\sin(m\psi)\, f^{(n)}(r,\mu) \end{aligned}\right\} \qquad (33)$$

$$\left.\begin{aligned} \phi^{(m,n,1,2)} &= P_n^m(\xi)\cos(m\psi)\, f^{(n)*}(r,\mu) \\ \phi^{(m,n,2,2)} &= P_n^m(\xi)\sin(m\psi)\, f^{(n)*}(r,\mu) \end{aligned}\right\} \qquad (34)$$

where f is given by Eqs. (29), (30), and (31), and * denotes the conjugate complex. Solutions with last subscript 2 represents incoming waves. One notices that $\phi^{(m,n,i,1)}$ is the conjugate complex of $\phi^{(m,\, n,\, i,\, 2)}$. There exist orthogonality relations

$$\int_{-1}^{+1} P_n^m(\xi)\, P_k^m(\xi)\, d\xi = 0 \quad ; \quad n \neq k$$

There exist also formulae for the normalization constant, but they are not needed here. Besides, one has orthogonality relations between the trigonometric functions sin $m\psi$ or cos $m\psi$. Then, one has for constant r

$$\iint \phi^{(m_1,n_1,i_1,k_1)}\, \phi^{(m_2,n_2,i_2,k_2)}\, do = 0 \qquad \text{for } (m_1,n_1,i_1) \neq (m_2,n_2,i_2)$$

where do is the surface element of the sphere. Notice that one may have $k_1 = k_2$. The derivative of the function $\phi^{(m,n,i,k)}$ with respect to r is the derivative with respect to the normal of this surface, it will be denoted by $d\phi/dn$. Now consider

$$\iint \left[\phi^{(m_1,n_1,i_1,k_1)} \frac{d}{dn}\phi^{(m_2,n_2,i_2,k_2)*} - \frac{d}{dn}\phi^{(m_1,n_1,i_1,k_1)}\, \phi^{(m_2,n_2,i_2,k_2)*}\right] do$$

This expression obviously vanishes for $(m_1, n_1, i_1) \neq (m_2, n_2, i_2)$. If these subscripts agree, and k_1 and k_2 are different, $k_1 = 1$ and $k_2 = 2$, say, then one obtains

$$\iint P_n^m(\xi)^2 \begin{Bmatrix} \sin^2(m\psi) \\ \cos^2(m\psi) \end{Bmatrix} r^2 \left[f^{(n)}(r)\, \frac{df^{(n)}(r)}{dr} - \frac{df^{(n)}}{dr}\, f(r)\right] d\tau$$

where $\partial\tau$ is the spatial angle of the surface element. If $k_1 = k_2 = 1$, then one obtains

$$\iint P_n^m(\xi^2) \begin{Bmatrix} \sin^2(m\psi) \\ \cos^2(m\psi) \end{Bmatrix} r^2 \left[f^{(n)}(r)\, \frac{df^{(n)}(r)*}{dr} - \frac{df^{(n)}}{dr}\, f^{(n)*}(r)\right] d\tau$$

The expression in the bracket is the Wronskian of two linearly dependent solutions of Eq. (28). Because of the factor $2/r$ of the term $df^{(n)}/dr$ in this equation, the Wronskian is given by const r^{-2}. This shows that the integral is independent of r. Thus, one has

$$\iint [\phi^{(m_1, n_1, i_1, k_1)} \frac{d}{dn} \phi^{(m_2, n_2, i_2, k_2)*} - \frac{d}{dn} \phi^{(m_1, n_1, i_1, k_1)} \phi^{(m_2, n_2, i_2, k_2)*}] do \tag{35}$$

$$= 0, \quad \text{if } (m_1, n_1, i_1, k_1) \neq (m_2, n_2, i_2, k_2)$$

If all superscripts are the same, then one obtains a constant independent of r. So far this formula holds only for surfaces r = const. In Section IV it will be extended to surfaces of arbitrary shape. The formula can be used to decompose a given function ϕ into incoming and outgoing waves.

SECTION III

INTERPRETATION OF THE SOLUTIONS DERIVED IN SECTION II

We start with the equation for the small acoustical perturbations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - a^{-2}\phi_{tt} = 0 \tag{36}$$

Let r and r' be vectors with components x, y, z and x', y', z', respectively, and let

$$r = |r - r'| \tag{37}$$

Then, one has the well known particular solutions

$$\phi(r, r', t) = r^{-1}(f(r-at)) \tag{38}$$

where f is an arbitrary function. Taking for f a function which has finite support (that is a function which differs from zero only in a finite interval) one clearly obtains the picture of a wave package which travels with the speed a in the direction of increasing values of r. (The shape of this package changes because of factor r^{-1}.) The expression (38) can be interpreted as the field caused by a source at the point $\vec{r} = \vec{r}'$, its strength is given by the momentary mass flux through a small sphere around this point. Assuming the density to be 1, one obtains the source strength at time t

$$-4\pi\, f(-at)$$

The total mass flow during a time interval from t_1 to t_2 is given by

$$-4\pi \int_{t_1}^{t_2} f(-at)dt$$

Choosing for f a delta function, one obtains the effect of a momentary source at (at time t')

$$f(-at) = \delta(-a(t-t'))$$

or

$$f(v) = \delta(v+at')$$

The total mass flow of this source is given by

$$-4\pi \int_{t=t'-\varepsilon}^{t=t'+\varepsilon} \delta(-a(t-t'))dt = 4\pi a^{-1} \int_{v=+a\varepsilon}^{v=-a\varepsilon} \delta(v)dv = -4\pi a^{-1}$$

The potential for a wave caused by a momentary source of strength 1 at time t = t' and at a point $\vec{r} = \vec{r}\,'$, is therefore given by

$$\phi(\vec{r}, \vec{r}\,', t) = -(4\pi)^{-1} a r^{-1} \delta(r - a(t-t')) \tag{39}$$

A flow field which can be represented as a superposition of such waves (with different times t' and different centers r') and where r' is confined to a finite region, satisfies the condition that at a sufficient distance it consists only of outgoing waves. This is our basic definition. Now consider a superposition of such waves whose strength is given by exp(iνt'). Then, with Eq. (39),

$$\phi(\vec{r}, \vec{r}\,', t) = -(4\pi)^{-1} a r^{-1} \int_{-\infty}^{t} \exp(i\nu t')\delta(r - a(t-t'))dt'$$

Let

$$v = r - a(t-t') \ , dv = a\,dt'$$

The integrand is zero except for v = 0, because of the presence of the delta function, that is for

$$t' = t - \frac{r}{a}$$

One thus obtains

$$\phi(\vec{r}, \vec{r}\,', t) = -(4\pi)^{-1} r^{-1} \exp(i\nu(t - a^{-1}r)) \tag{40}$$

This result can be obtained directly from Eq. (38).

The equation for an unsteady flow linearized for the vicinity of a parallel flow in the x direction with velocity U arises from Eq. (36), if one views the flow field from a coordinate system traveling with velocity U in the negative x direction. Setting

$$\bar{x} = x + Ut, \ \bar{y} = y, \ \bar{z} = z, \ \bar{t} = t \tag{41}$$

$$\phi(x, y, z, t) = \bar{\phi}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$$

one arrives at Eq.(1).

We now derive particular solutions of Eq. (1) by a superposition of momentary sources of strength $\exp(i\nu t)$ moving in the negative x direction with velocity U along a line y = y' = const, z = z' = const so that it arrives at the station $\bar{x}$ at time t' = 0. The flow field so arising is viewed from a system of coordinates which moves with the same speed in the negative x direction so that the origin of the moving system of coordinates arrives at the origin of the system of coordinates fixed in the air at rest t time 0. In the moving system the source lies at $\bar{x} = \bar{x}'$ at all times. Accordingly, we chose

$$\vec{r}\,' = [\bar{x}' - Ut', y', z']^+$$
$$r = [\bar{x} - Ut, \bar{y}, \bar{z}]^+ \tag{42}$$

Let

$$\rho^2 = (y-y')^2 + (z-z')^2 = (\bar{y}-y')^2 + (\bar{z}-\bar{z}')^2 \tag{43}$$

Then one has

$$r = |\vec{r} - \vec{r}\,'| = [((\bar{x}-\bar{x}') - U(t-t'))^2 + \rho^2]^{1/2} \tag{44}$$

Using Eq. (39), one then obtains for the potential in a moving system of coordinates

$$\phi(\bar{r}, \bar{r}', t) = -a(4\pi)^{-1} \int_{-\infty}^{t} \exp(i\nu t')\, r^{-1} \delta(r - a(t-t'))\, dt' \tag{45}$$

with

$$\bar{r} = (\bar{x}, \bar{y}, \bar{z})^+$$
$$\bar{r}' = (\bar{x}', \bar{y}', \bar{z}')^+$$

We set

$$v = r - a(t-t') = [((\bar{x}-\bar{x}') - U(t-t'))^2 + \rho^2]^{\frac{1}{2}} - a(t-t') \tag{46}$$

then

$$dv = [r^{-1}((x-\bar{x}') - U(t-t'))U + a]\,dt'$$

Hence

$$\phi(\bar{r}, \bar{r}', t) = -a(4\pi)^{-1} \int_{-\infty}^{t} \exp(i\nu t')[U(x-x') - U^2(t-t') + ar]^{-1} \delta(v)\, dv \tag{47}$$

The integrand is zero except for v = 0. Hence, from Eq. (46)

$$t-t' = -\frac{U(\bar{x}-\bar{x}')}{a^2-U^2} + \frac{a}{a^2-U^2}\left[(\bar{x}-\bar{x}')^2+\left(1-\frac{U^2}{a^2}\right)\rho^2\right]^{1/2} \tag{48}$$

Only the positive root appears because of the limits of the integral, Eq. (45). For v = 0 one obtains from Eq. (46)

$$r = a(t - t')$$

One obtains for the term in the bracket of Eq. (47)

$$U(\bar{x}-\bar{x}') - U^2(t-t') + ar = a\left[(\bar{x}-\bar{x}')^2+\left(1-\frac{U^2}{a^2}\right)\rho^2\right]^{1/2}$$

Using Eq. (48), we finally obtain

$$\varphi(\bar{r},\bar{r}',t) = -(4\pi)^{-1}\exp(i\nu t)\left[\exp\left(\frac{i\nu U(\bar{x}-\bar{x}')}{a^2-U^2}\right)\right]\left[\exp\left(-\frac{i\nu a}{a^2-U^2}\left[(\bar{x}-\bar{x}')^2+(1-M^2)(\bar{y}^2+\bar{z}^2)\right]\right)\right]\times$$
$$\times\left[(\bar{x}-\bar{x}')^2+(1-M^2)(\bar{y}^2+\bar{z}^2)\right]^{-1/2}$$

This expression is singular at the point $\bar{r} = \bar{r}'$. It leads directly to the particular solution, Eq. (10). This shows that this particular solution represents the potential of an oscillating source moving in air at rest. Thus, the fundamental solution, Eq. (10), represents a superposition of outgoing acoustical waves.

The corresponding solutions, Eq. (18), can be obtained by the so called method of descent; that is, one considers a distribution of sources of the form, Eq. (10), which is constant along a line $\bar{x} = \bar{x}'$, $\bar{y} = y'$ of the three dimensional space. The potential is then obtained by an integration with respect to $\bar{z}'$, the result is independent of $\bar{z}$. One obtains in this manner one of the integral representations of the Hankel function.

Differentiating the three dimensional fundamental solution Eq. (10), with respect to either x', y' or z' or equivalently with respect to x, y, or z, n times one obtains expressions of the form, Eq. (29), in which the functions $\hat{f}$ are polynomials of the degree n. Such expressions satisfy the original Helmholtz Eq. (7).

Not all of these derivatives are independent because the function to be differentiated satisfies the differential Eq. (7). The process of differentiation is the limit of a superposition of expressions (10). Accordingly, these derivatives amount to a superposition of outgoing acoustical waves.

If one carries out a fixed number of n_1 of such differentiations, then one obtains a finite number of linearly independent expressions of the form

$$\phi = S(\psi, \theta) f(r)$$

where the function f(r) has the form

$$f(r) = \exp(-i\mu r)\, \tilde{P}(r)\, r^{-n_1-1}$$

in which the function $\tilde{P}(r)$ is a polynomial of degree n_1.

These particular solutions can be represented by the $\phi^{m,n,i,k}$ defined in Eq. (33). At infinity the solutions with $\exp(-i\mu r)$ and $\exp(+i\mu r)$ (see the definition of f in Eq.(39)) behave as r^{-1}. Therefore, one must admit functions $\phi^{m,n,i,k}$ with superscript k = 2 as well as k = 1. Since at the origin one of these derivatives has as lowest powers r^{-n_1-1}, one needs to admit particular solutions ϕ with $m \leq n \leq n_1$. There are only a finite number of such solutions. Then one has the following situation. Consider ψ and θ as fixed. Along such a ray through the origin the function ϕ obtained by forming the n_1^{th} mixed derivative has the form

$$r^{-n_1-1} \exp(-i\mu r)\, \tilde{P}^{(1)}_{n_1}(r)$$

The particular solutions (Eq . (33)) used for the representation of this function combine into an expression of the form

$$r^{-n_1-1}\, \tilde{P}^{(2)}_{n_1} \exp(-i\mu r) + r^{-n_1-1}\, \tilde{P}^{(3)}_{n_1} \exp(i\mu r)$$

Equating these two expressions, one obtains

$$r^{-n_1-1}\, \tilde{P}^{(1)}_{n_1}(r) \exp(-i\mu r) = r^{-n_1-1}\, \tilde{P}^{(2)}_{n_1} (\exp(-i\mu r) + r^{-n_1-1}\, \tilde{P}^{(3)}_{n_1} \exp(+i\mu r)$$

and hence

$$\tilde{P}^{(1)}_{n_1} - \tilde{P}^{(2)}_{n_1} = \tilde{P}^{(3)}_{n_1} \exp(2i\mu r)$$

Now one has a polynomial in r on the left and a transcendental function of r on the right. This equation can be satisfied only if $P^{(3)}_{n_1} = 0$. In other words, the representation of the derivatives contains only functions $\phi^{(m,n,i,k)}$ in which the last subscript is k = 1. Now it is a matter of counting the number of linearly independent derivatives and the number of functions $\phi^{(m,n,i,1)}$ in order to demonstrate that the relations so constructed can be inversed; in other words, that the functions $\phi^{(m,n,i,1)}$ can be represented as a linear combination of derivatives of the fundamental solution (Eq. (10)). Then it follows that the expressions (Eq. (33)) with last subscript 1 can be interpreted as a superposition of outgoing acoustic waves.

Specific formula are not needed in the present context. They are derived in the appendix.

Now one has a polynomial in r on the left and a transcendental function of r on the right. This equation can be satisfied only if $P^{(3)}_{n_1} = 0$. In other words, the representation of the derivatives contains only functions $\phi^{(m,n,i,k)}$ in which the last subscript is $k = 1$. Now it is a matter of counting the number of linearly independent derivatives and the number of functions $\phi^{(m,n,i,1)}$ in order to demonstrate that the relations so constructed can be inversed; in other words, that the functions $\phi^{(m,n,i,1)}$ can be represented as a linear combination of derivatives of the fundamental solution (Eq. (10)). Then it follows that the expressions (Eq. (33)) with last subscript 1 can be interpreted as a superposition of outgoing acoustic waves.

Specific formula are not needed in the present context. They are derived in the appendix.

SECTION IV
GREEN'S THEOREM

So far we have derived a system of particular solutions of the Helmholtz equation including a fundamental solution and we have shown that they can be interpreted as a superposition of outgoing acoustical waves. These particular solutions satisfy the far field conditions. Flow fields which can be represented as a superposition of such expressions will also satisfy the far field conditions. This statement can be regarded as a characterization of such flow fields. In a further development, we bring this criterion into a form which is more practical. This is done by means of Green's formula.

Let

$$L(\phi) = \phi_{xx} + \phi_{yy} + \phi_{zz} + \mu^2\phi \tag{49}$$

and let ϕ and ω satisfy

$$L(\phi) - h(\vec{r}) = 0 \tag{50}$$

$$L(\omega) = 0 \tag{51}$$

where $h(\vec{r})$ is considered as known. Let R be some region in the x,y,z space and ∂R its boundary. Within this region Eqs. (50) and (51) are satisfied everywhere. Then one has, because of Eq. (51)

$$\iiint_R [L(\phi(\vec{r}')) - h(\vec{r}')]\,\omega(\vec{r}')\,dv' = 0$$

where dv' is the volume element in the three dimensional space of the variable x',y',z'. Denote by $\psi_{n'}$ and $\omega_{n'}$ the derivative in the direction of the outer normal in the x', y', z' space. Then one has the familiar formula

$$\iint_{\partial R} [\omega(\vec{r}')\phi_{n'}(\vec{r}') - \omega_{n'}(\vec{r}')\phi_{n'}(\vec{r}')\,do' - \iiint_R h(\vec{r}')\,\omega(\vec{r}')\,dv' = 0$$

Taking for $\omega(\vec{r}')$ the expression $\omega(\vec{r},\vec{r}')$ given in Eq. (10) surrounding the singular point $\vec{r} = \vec{r}'$ by a small sphere in the $\vec{r}'$ space, one obtains in a familiar manner

$$\phi(\vec{r}) = \iint_{\partial R} [-\omega(\vec{r},\vec{r}')\phi_{n'}(\vec{r}') + \omega_{n'}(\vec{r},\vec{r}')\phi(\vec{r}')]\,do' + \iiint_{R} h(\vec{r}')\omega(\vec{r},\vec{r}')\,dv' \tag{52}$$

For our application, one considers a region bounded by the two surfaces ∂R_1 and ∂R_2. The normal derivatives always refers to the outer normal of the region R. One is not justified to disregard the contribution of the surface ∂R_2 even if it should move to infinity, for the expressions ψ and ω do not die out sufficiently fast. The limiting process is justified for the steady case ($\mu = 0$). Eq. (52) is an identity. It holds only if the values of $\phi_{n'}(\vec{r}')$ and $\phi(\vec{r}')$ are taken from a function ϕ which satisfies Eq. (50). For a further discussion, consider an expression

$$\phi(\vec{r}) = \iint_{\partial R} [-\omega(\vec{r},\vec{r}')f_1(\vec{r}') + \omega_{n'}(\vec{r},\vec{r}')f_2(\vec{r}')]\,do' + \iiint_{R} h(\vec{r}')\omega(\vec{r},\vec{r}')\,dv' \tag{53}$$

The function $\omega(r,r')$ satisfies

$$L_{\vec{r}}(\omega(\vec{r},\vec{r}') = 0$$

where the subscript r under L indicates that the operation L is to be carried out with respect to the variable $\vec{r}$, while $\vec{r}'$ is a fixed parameter. Because of the normalization of $\omega(r,r')$, one then finds from the last formula that

$$L(\phi(\vec{r})) = h(\vec{r}) \tag{54}$$

everywhere inside and R and

$$L(\phi(\vec{r})) = 0 \tag{55}$$

outside the region. For small $r = |\vec{r} - \vec{r}'|$ the function $\omega(r,r')$ behaves as $-4\pi/r$; that is, exactly as the fundamental solution of the Laplace equation. The surface integrals, therefore represent a layer of poles and dipoles with densities given respectively by $f_1(r')$ and $f_2(r')$. A layer of poles generates a function ϕ which is continuous everywhere and for which the normal derivative has a jump given by the density of the poles, here by f_1. A layer of dipoles oriented in the direction of the normal gives a function ϕ for which the normal derivative is continuous everywhere and for which the jump of the potential is the negative of the dipole density ($-f_2$). This is an interpretation of Eq. (53). If f_1 happens to be equal to the values of ϕ_n pertaining to an actual solution and $-f_2$ happens to be the value of ϕ, then the potential outside of R is identically equal to zero and one recovers inside the function $\phi(\vec{r})$.

Let ∂R be the contour of a body. An expression

$$\phi(\vec{r}) = \iint_{\partial R} [-\omega(\vec{r},\vec{r}')f_1(\vec{r}') + \omega_{n'}(\vec{r},\vec{r}')f_2(\vec{r}')]do' + \iiint_R h(\vec{r}')\omega(\vec{r},\vec{r}')dv'$$

satisfies

$$L(\phi) = h(\vec{r})$$

and the far field condition for ϕ, for it is a linear combination of functions which satisfy this condition. If $f_2(r')$ and $f_1(r')$ happen to be identical with ϕ and its normal derivative at ∂R, then the above formula gives a representation for ϕ in the field. (Usually ϕ_n is prescribed and ϕ is unknown.) Notice that in this derivation, we have used results of potential theory rather than Green's formula. A comparison with Eq. (52) shows that the integral over the outer boundary does not appear. How this comes about is seen by the following argument.

Let $h(r') = 0$. Then one has

$$\iint_{\partial R_1} [-\phi_{n'}(\vec{r}')\omega(\vec{r}') + \phi(\vec{r}')\omega_{n'}(\vec{r}')]do' + \iint_{\partial R_2} [-\phi_{n'}(\vec{r}')\omega(\vec{r}') + \phi(\vec{r}')\omega_{n'}(\vec{r}')]do'$$

This shows that the integral

$$\iint_{\partial R} [-\phi_{n'}(\vec{r}')\omega(\vec{r}') + \phi(\vec{r}')\omega_{n'}(\vec{r}')]do'$$

remains unchanged if one carries out a continuous deformation of the surface ∂R through a region in which

$$L(\varphi) = 0 \quad \text{and} \quad L(\omega) = 0$$

Notice that ϕ or ω need not satisfy special far field conditions. Such a deformation can, for instance, be carried out for the spherical surface over which the integration in Eq. (34) is carried out. We apply this result to two functions $\omega(r,r')$ and $\omega(r'',r')$

where the singular points r' = r and r' = r" lie inside of the contour ∂R

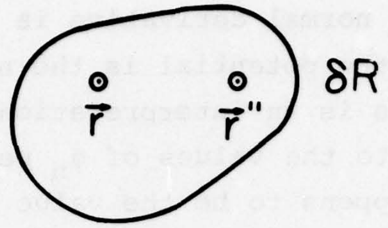


Figure 2. Surface ∂R in the r' - space.

The surface ∂R can be contracted into two small spheres around the singular points $\vec{r}' = \vec{r}$ and $\vec{r}' = \vec{r}''$, and then the integrals can be evaluated. One finds that the contributions of the two spheres cancel each other. Thus,

$$\iint_{\partial R} [\omega(\vec{r},\vec{r}')\,\omega_{n'}(\vec{r}'',\vec{r}') - \omega_{n'}(\vec{r},\vec{r}')\,\omega(\vec{r}'',\vec{r}')]\,do' = 0 \tag{56}$$

A function $\phi(r)$ that satisfies the far field condition can be represented as

$$\phi(r) = \sum_{\ell} \beta_{\ell}\,\omega(\vec{r},\vec{r}_{\ell}'')$$

This function evaluated at r = r' gives

$$\phi(r') = \sum_{\ell} \beta_{\ell}\,\omega(\vec{r}',\vec{r}_{\ell}'')$$

Thus, one finds that

$$\iint_{\partial R} [\omega(\vec{r},\vec{r}')\,\phi_{n'}(r') - \omega_{n'}(\vec{r},\vec{r}')\,\phi(\vec{r}')]\,do' = 0 \tag{57}$$

This explains why the integral over the outer contour is not present in Eq. (52), if the far field conditions are satisfied. It is, of course, assumed that h(r') = 0 outside of the outer contour, for Eq. (57) does not hold if one of the singular points r' = r or r' = r" lies outside of the surface of integration.

The relation (Eq. (57)) constitutes a far field criterion. It must be satisfied for every choice of r (although one will then obtain redundant conditions). It therefore holds also for derivatives of this equation with respect to r, that is, if the function ω(r,r') is replaced by its derivatives with respect to x, y, or z, usually at the origin. But these derivatives can be expressed by the functions $\phi^{(n, m, i, 1)}$ (Eq. (33)). Thus, one obtains a further relation

$$\iint_{\partial R} [\phi^{(m,n,i,1)}(\vec{r}')\phi_{n'}(\vec{r}') - \phi_{n'}^{(m,n,i,1)}(\vec{r}')\phi(\vec{r}')]do' = 0 \tag{58}$$

for all functions $\phi^{(m,n,i,1)}$ and an arbitrary outer boundary surface of the region under consideration. One can gain from Eq. (57) another form of the far field criterion. Let S be a surface in the flow field which includes all points at which $h(r') \neq 0$ but lies inside of the boundary ∂R of the computed field. Let ϕ be the solution of a problem

$$L(\phi(\vec{r})) - h(\vec{r}) = 0$$

which satisfies the far field conditions and let ϕ_S be the values of ϕ at the surface S. Then one can solve the Dirichlet problem, which has $L\phi = 0$ inside of S (including the region within an inner bounding body surface). This can always be done provided that μ^2 is not an eigenvalue. This means that $L\phi = 0$ with the boundary condition $\phi = 0$ at S does not have a nontrivial solution. Excluding this possibility, one has now a function ϕ which is represented outside of S by the solution ϕ of the original problem satisfying the far field condition and inside of S by the solution of the Dirichlet problem just described. At S, this function ϕ is continuous, but it has a jump of the normal derivative. This flow field can then be represented by a layer of poles distributed over S where the pole density is given by the jump of the normal derivative. Thus, we have, as the representation for the original ϕ outside of S

$$\phi(\vec{r}) = \iint_S f(\vec{r}')\,\omega(\vec{r},\vec{r}')\,do', \quad \vec{r}' \subset S \tag{59}$$

This representation can be applied in particular to the outer surface ∂R where the farfield conditions are to be formulated. Here the function ϕ is represented by a superposition of particular solutions $\omega(r,r')$ which satisfy the far field conditions and where the singular points are arranged at a surface S.

The limiting case where the surface S approaches the outer boundary of the computed flow field is admissible (and sometimes practical). In cases where μ^2 is an eigenvalue of the problem for the function S, or even close to an eigenvalue, one will find that the function f in Eq. (59) becomes very large. For numerical reasons, such cases must be excluded.

SECTION V
FAR FIELD CRITERIA

We started from the definition that any superposition of expressions which ultimately can be considered as a linear combination of expression representing outgoing acoustical waves satisfies the far field conditions. From this idea the following criteria have been derived.

1. The solution is a superposition of expressions (33)
2. The solution satisfies the condition Eq. (58)
3. The solutions can be represented in the form of Eq. (59)
4. The Klunker-Traci formulation

Formulations 1 and 3 are closely related to each other. In each case the solution at the outer boundary of the computed flow field, (this includes ϕ_n as well as ϕ) is represented by a linear combination of particular solutions which satisfy the far field conditions.

In the formulation 1, there arises the question of convergence if the outer surface of the region under consideration deviates strongly from a circle.

In the formulation 3, one has the restriction that μ^2 must not be an eigenvalue of the Dirichlet problem for the chosen surface S. We shall see in an example that this may indeed be an impediment. In the formulation 2, the convergence problem appears in a different form. The functions ω play the role of a test function. If the value of r varies rather strongly along the outer surface, and if n is large, then because of the character of the function $f^{(n)}(r)$, the functions are large for small values of r and, comparatively small for large values of r. For large values of n, the particular solutions behave roughly as r^{-n}. This means that the particular solutions ϕ become nearly linearly dependent for large values of n , if r varies strongly along the outer surface.

We shall show in an example that the exclusion of the vicinity of eigenvalues in the application of the criterion 3 may be necessary. If μ is an eigenvalue, then there exists one or more functions ϕ for which $L\phi = 0$ in the interior of S and for which $\phi = 0$ at the surface S. The normal derivative of this function in the interior is, of course, not zero. This function can be continued outside of S by $\phi = 0$. A layer of single poles with density given by the values of ϕ_n pertaining to the eigenfunction will represent this function ϕ and therefore give $\phi = 0$ outside. In other words, one has one or more density distributions which will not contribute to the functions ϕ and ϕ_n at the outer edge of the computed flow field. We shall see in an example that the particular solutions for ϕ which are lost because of this phenomenon are needed.

The formulation 2 of the far field conditions is particularly useful if one has to make a refinement to an existing approximate solution and one knows that the contributions of long waves are small. (These are the solutions for which one may find oneself in the vicinity of an eigenvalue.) The criterion 2 works rather well for long waves (low values of n) and the criterion 3 for short waves (large values of n). In this sense the criteria 2 and 3 are complementary. Further remarks about the application of this criterion will be made later.

The formulation of Klunker and Traci is based on Eq. (53). Here f_1 is the given normal component of ϕ and f_2 the unknown potential. In essence, this criterion is rather similar to criterion 3 shown above except that the surface S is contracted to the body surface and that as a consequence, one takes the presence of sources h(r') within the field into account.

There is, however, a difference. In the formulation 3, the function f is considered as unknown and one expresses ϕ and ϕ_n at the outer edge of the flow field in terms of this function. In the final computed flow field it must be possible to represent ϕ_n as well as ϕ by means of a suitably chosen function f. The same can, of course, be done with the use of Klunker's formulation. In this case the potential at the surface plays the role of the function f, but one does not make direct use of the fact that this function has this specific interpretation.

In the Klunker-Traci formulation one proceeds, however, in a different manner. One uses the expression (Eq. (52)) to compute only ϕ at the outer boundary. Next, one uses these values of ϕ as the far field condition and computes the flow field. In this computation, one finds the surface potential ϕ. One has the desired solution if the surface potential so found agrees with the potential used to compute ϕ at the far field boundary.

The difference between the formulation 3 and 4 lies in the fact that in formulation 4 one disregards the condition for ϕ_n at the far boundary which one could obtain from Eq. (52), and replaces it by the requirement that the function f_2 which appears in Eq. (53) is identical with the surface potential obtained from the flow field computation. The Klunker Traci formulation is well suited to an iterative approach in which one updates the surface potential in each iteration step. It becomes inconvenient if iterations should fail.

SECTION VI
AN EXAMPLE

The following example shows how these conditions can be applied. It is extremely simple; in fact, all steps which ordinarily require numerical techniques are carried out analytically. The sole purpose is to show which steps must be taken if one applies different criteria. We shall consider solutions of the Helmholtz equation, interpreted as a description of the propagation of small perturbations in air at rest. The boundary of the body is given by a sphere which pulsates harmonically in time. The three dimensional case has been chosen because of the simplicity of the fundamental solution. The field has spherical symmetry because the boundary conditions are assumed to have spherical symmetry. Accordingly, we deal with a one dimensional problem. It is, of course, important that the steps for which we use analytical formula must, in a realistic case, be carried out by numerical techniques in one or three dimensions. While in the present case, the far field conditions depend upon only one parameter, one will have in reality a great number of such parameters.

The field for which the computations are carried out lies between the radius r_0 (radius of the pulsating sphere) and a radius r_1, at which the far field conditions are to be applied. At $r = r_0$ the value of $\phi_r = \phi_{r0}$ = const is prescribed. The differential equation to be satisfied reduces to

$$\phi_{rr} + 2r^{-1}\phi_r + \mu^2\phi = 0 \tag{60}$$

The fundamental solution which satisfies the radiation condition is given by

$$\phi = -(4\pi)^{-1} r^{-1} \exp(-i\mu r) \tag{61}$$

The problem has the following solution

$$\phi = \frac{-r_0\phi_{r0}}{1+i\mu r_0}\,\frac{r_0}{r}\exp[-i\mu(r-r_0)] \tag{62}$$

It is readily verified that it satisfies the boundary condition

$$\phi_r\big|_{r=r_o} = \phi_{r_o}$$

Because of the spherical symmetry of the problem, the density functions for the distribution of poles or dipoles occurring in various formulae will also be spherically symmetric. At various occasions, for instance in the expression (50) we shall encounter expressions $\phi(r) = \iint F(\vec{r}')\ \omega\ (\vec{r}-\vec{r}')do'$, where the integration is to be extended over a sphere r' = const and where $F(r')$ = const. Thus, one has to evaluate integrals

$$\phi'(\vec{r}) = \iint \omega(\vec{r}-\vec{r}')do'$$

The numerical evaluation which is usually needed is replaced here by an analytical approach in which we construct functions which have the same properties as these integrals. To be specific, the functions which arise by these integrals satisfy the Helmholtz equation inside and outside of the spherical layer. At infinity they satisfy the radiation condition and in case of a layer of single poles with density 1, one has continuity of the potential and a jump of the derivative normal to the layer. In the case of a dipole distribution, one has continuity of the normal derivative and a jump of -1 of the potential. Let the potential of a single layer at radius r_o be given by $\phi^{(1)}$, and the potential of a layer of dipoles by $\phi^{(2)}$. The direction of the outer normal which must be taken at $r = r_o$ is the negative r direction. Then one has

$$\begin{aligned} \phi^{(1)} &= -\frac{1}{i\mu}\frac{r_o}{r}\exp(-i\mu r_o)\sin\mu r\ , \quad r < r_o \\ \phi^{(1)} &= -\frac{1}{i\mu}\frac{r_o}{r}\exp(-i\mu r)\sin\mu r_o\ , \quad r > r_o \end{aligned} \tag{63}$$

One verifies that

$$\phi^{(1)}(r_o+0) - \phi^{(1)}(r_o-0) = 0$$

$$\phi_r^{(1)}(r_o+0) - \phi_r^{(1)}(r_o-0) = 1$$

Moreover,

$$\phi^{(2)}(r) = -\frac{1+i\mu r_0}{\mu r_0}\,\frac{r_0}{r}\,\exp(-i\mu r_0)\,\sin\mu r \quad ; \quad r < r_0 \tag{64}$$

$$\phi^{(2)}(r) = [\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}]\,\frac{r_0}{r}\,\exp(-i\mu r) \quad ; \quad r > r_0$$

One verifies that

$$\phi^{(2)}(r_0+0) - \phi^{(2)}(r_0-0) = 1$$

$$\phi^{(2)}_n(r_0+0) - \phi^{(2)}_n(r_0-0) = \phi^{(2)}_r(r_0-0) - \phi^{(2)}_r(r_0+0) = 0$$

Also needed in the analysis is a general solution which satisfies the boundary conditions at the surface of the body. In a numerical approach, one usually does not encounter such an expression directly. The analytical formulation is used to replace information usually obtained by integrating the flow equations. One has

$$\phi = \frac{-r_0\phi_{r0}}{1+i\mu r_0}\,\frac{r_0}{r}\,\exp(-i\mu(r-r_0)) + C\,\frac{r_0}{r}\,[\sin[\mu(r-r_0)] + \mu r_0\cos[\mu(r-r_0)]] \tag{65}$$

The first part is the analytic solution of the problem (which satisfies the inhomogeneous boundary conditions at $r = r_0$, the second part has an r derivative 0 at $r = r_0$.

After these preparations, we can apply different far field conditions. In the formulation one expresses the far field (that is the field at $r = r_1$) by a superposition of solutions (33). Because of the spherical symmetry of the present problem, only one of the solutions, Eq. (33) namely,

$$\phi^{(1,1,1,1)} = -(4\pi)^{-1}\,r^{-1}\exp(-i\mu r)$$

is encountered, with a coefficient c_1, say.

Then one has at $r = r_1$

$$\phi(r_1) = -c_1(4\pi)^{-1}\,r_1^{-1}\exp(-i\mu r_1)$$

$$\phi_n(r_1) = c_1(4\pi)^{-1}\left(\frac{1}{r_1^2} + \frac{i\mu}{r_1}\right)\exp(-i\mu r_1) \tag{66}$$

The expression Eq. (65) gives

$$\phi(r_1) = \frac{-r_0\phi_{r0}}{1+i\mu r_0}\,\frac{r_0}{r_1}\,\exp(-i\mu(r_1-r_0)) + C\,\frac{r_0}{r_1}\,[\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))] \tag{67}$$

$$\phi_r(r_1) = \frac{-r_0\phi_{r_0}}{1+i\mu r_0}\left(-\frac{r_0}{r_1^2} - i\mu\frac{r_0}{r_1}\right)\exp(-i\mu(r_1-r_0))$$

$$+ C\left\{-\frac{r_0}{r_1^2}[\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))] + \frac{r_0}{r_1}\mu[\cos(\mu(r_1-r_0)) - \mu r_0\sin(\mu(r_1-r_0))]\right\} \quad (68)$$

Usually, one avoids the construction of a general solution by numerical solution of the difference equations. (This would require far too many free parameters.) In the present problem, in difference form, one obtains a linear system for the values of ϕ at the points of a grid. If one eliminates the unknowns from this linear system, starting at the profile (here $r = r_0$) and proceeding toward the outer boundary one finally obtains relations between the values of ϕ at the outer boundary and at the row of grid points next to it. These relations are equivalent to a linear relation between ϕ and ϕ_n at the outer boundary. In the present case, the corresponding information would be obtained by eliminating C from Eqs. (67) and (68). Into this equation. one then substitutes the expressions Eq. (66) which express the far field conditions. One thus obtains an equation for C_1. The solution can obviously be found by inspection. It is given by Eqs. (67) and (68) with C = 0.

In the second formulation, one uses Eq. (58) because of the spherical symmetry one needs to consider only the one function $\phi^{(1,1,1,1)}$. This gives the relation

$$-(4\pi)^{-1}\frac{1}{r_1}\exp(-i\mu r_1)\phi_n(r_1) - (4\pi)^{-1}\frac{1}{r_1^2}(1+i\mu r_1)\exp(-i\mu r_1)\phi(r_1) = 0 \quad (69)$$

This is obviously equivalent with Eqs. (66). This equation is now combined with the relation between ϕ_n and ϕ obtained from the flow field (Eqs. (67) and (68) with C eliminated). The problem is, of course, very similar to the previous case, except that the constant C_1 does not occur. One has in Eq. (69) a direct relation between the values of ϕ and ϕ_n at the outer contour.

The third form of the criterion gives, for the present example, the same procedure as the first one since there is only one function, $\omega(\vec{r},\vec{r}')$ which is spherically symmetric and satisfies the far field conditions.

We describe the fourth criterion (Klunker-Traci) in a manner which is suited for a noniterative approach. First, one expresses, by means of Green's formula, the values of ϕ and ϕ_r at $r = r_1$ in terms of the given function $\phi(r_o)$ and of the unknown surface potential $\phi(r_o)$ here denoted by f_1. Making a token evaluation of Eq. (52) (actually using Eqs. (63) and (64)) and remembering that at $r = r_o$, $\phi_n = -\phi_{r0}$, one obtains

$$\phi(r_1) = \left\{-\phi_{r0}\frac{1}{\mu}\sin(\mu r_0) + f_1\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right]\right\}\frac{r_0}{r_1}\exp(-i\mu r_1) \tag{70}$$

$$\phi_r(r_1) = \left\{-\phi_{r0}\frac{1}{\mu}\sin(\mu r_0) + f_1\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right]\right\}\left(-\frac{r_0}{r_1^2} - i\mu\frac{r_0}{r_1}\right)\exp(-i\mu r_1) \tag{71}$$

Taking the original form of the Klunker Traci formulation, we now construct a solution (in practice by solving the difference equation for the flow field) which assumes the values of $\phi(r_1)$ at the outer edge of the flow field. (Notice that this formulation still contains the unknown function f_1.) This is done here by choosing the constant C in the expression (67). It is useful to rewrite the last equation. We know that the ultimate solution is obtained for C = 0. The function f_1 pertaining to it is then given by

$$f_1 = -\frac{r_0\phi_{r0}}{1+i\mu r_0}$$

Now we set

$$f_1 = -\frac{r_0\phi_{r0}}{1+i\mu r_0} + \delta f_1 \tag{72}$$

Then one obtains from the boundary condition (70)

$$\phi(r_1) = \left\{-\phi_{r0}\frac{1}{\mu}\sin(\mu r_0) - \frac{r_0\phi_{r0}}{1+i\mu r_0}\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right] + \delta f_1\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right]\right\}\frac{r_0}{r_1}(\exp(-i\mu r_1)$$

Hence,

$$\phi(r_1) = -\frac{r_0\phi_{r0}}{1+i\mu r_0}\frac{r_0}{r_1}\exp(-i\mu(r_1-r_0)) + \delta f\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right]\frac{r_0}{r_1}\exp(-i\mu r_1) \quad (73)$$

The first term in this equation agrees, of course, with Eq. (62) and the first term in Eq. (65). To satisfy the boundary conditions, one now obtains the condition (from Eq. (65) and the last equation).

$$\delta f_1\left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right]\frac{r_0}{r_1}\exp(-i\mu r_1) = C\frac{r_0}{r_1}\left[\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))\right]$$

Hence,

$$C = \delta f_1 \frac{[\mu r_0\cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\mu r_0[\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))]} \quad (74)$$

Inserting this value of C into Eq. (65), one obtains an expression for ϕ which satisfies the boundary condition at $r = r_0$ and assumes the values of ϕ at $r = r_1$ given by Eq. (70). (The derivation shown here replaces a numerical integration for the flow field.) This solution still depends upon the unknown δf_1. From this solution, one obtains the potential at the surface $r = r_0$. One has from Eq. (65)

$$\phi(r_0) = \frac{-r_0\phi_{r0}}{1+i\mu r_0} + \delta f_1 \frac{[\mu r_0\cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\mu r_0[\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))]}\mu r_0$$

Now one has the requirement that the function f_1 originally assumed agree with the value of the potential at $r = r_0$ found from integrating the flow equation. Hence,

$$\frac{-r_0\phi_{r0}}{1+i\mu r_0} + \delta f_1 = \frac{-r_0\phi_{r0}}{1+i\mu r_0} + \delta f_1 \frac{[\mu r_0\cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))}$$

Hence

$$\delta f_1\left(1 - \frac{[\mu r_0\cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\sin(\mu(r_1-r_0)) + \mu r_0\cos(\mu(r_1-r_0))}\right) = 0$$

This is solved by $\delta f_1 = 0$ (as expected) unless the factor of δf_1 vanishes. In this factor some simplifications are possible. One ultimately obtains

$$\delta f_1 \frac{\sin(\mu r_1)\exp(-i\mu r_0)(1+i\mu r_0)}{\sin(\mu(r_1-r_0)) + \mu r_0 \cos(\mu(r_1-r_0))} = 0 \tag{75}$$

The procedure fails if $\sin(\mu r_1) = 0$. Then the condition is satisfied for any value of δf_1.

In an iterative procedure, one starts with an assumed value of ϕ at the surface $r = r_0$, that is with some choice of δf_1 and then computes by means of Green's formula the potential at $r = r_1$. According to Eq. (73), it deviates from the unknown potential of the exact solution by

$$\delta f_1 \left[\cos(\mu r_0) - \frac{\sin(\mu r_0)}{\mu r_0}\right] \frac{r_0}{r_1} \exp(-i\mu r_1)$$

With this boundary condition, one integrates the flow equation, and arrives at a surface potential whose deviations from the exact solution is given by

$$\delta f_1 \frac{[\mu r_0 \cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\sin(\mu(r_1-r_0)) + \mu r_0 \cos(\mu(r_1-r_0))}$$

The potential so obtained is then taken as the starting point of a new iteration step. The iterations converge (in this particular case) if the absolute value of δf_1 decreases, that is, if

$$\left| \frac{[\mu r_0 \cos(\mu r_0) - \sin(\mu r_0)]\exp(-i\mu r_1)}{\sin(\mu(r_1-r_0)) + \mu r_0 \cos(\mu(r_1-r_0))} \right| < 1$$

In general, this condition is not satisfied. It is, however, satisfied for μ sufficiently small. For $\mu = 0$ the correct value of ϕ is obtained in one step. The denominator in the last expression vanishes if μ^2 is an eigenvalue of the problem

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} + \mu^2 \varphi = 0$$

with boundary conditions $\phi_n = 0$ at the surface of the body and $\phi = 0$ at the outer contour. If one is in the vicinity of such an eigenvalue, and one tries to solve the equation for ϕ numerically (or analytically) then the solution will obviously become very large. A noniterative application of the Klunker-Traci procedure in this form is made difficult by the fact that one must first compute the flow field with conditions at the outer boundary containing the unknown surface potential, before one can apply the condition that the potential obtained from the flow field computation must equal the potential assumed in assigning the far field conditions. For noniterative applications, the far field condition 2 may be more suitable.

The nonconvergence of the iterative process can probably be avoided if one derives from the values of ϕ and ϕ_n determined by Green's formula (here Eqs. (70) and (71)) a different boundary condition. For the present problem, this possibility has been discussed by the author with the $-i\mu\phi + \phi_n$ number prescribed according to the values computed from Eqs. (70) and (71). In this formulation, the occurrence of the imaginary unit is important. One finds that the denominator of the last expression will not vanish. This would probably allow one to use an iterative procedure in conjunction with the Klunker-Traci method. However, the difficulty of nonconvergence still exists if one tries to carry out the flow field calculations for these boundary conditions by an iterative process.

For a case with spherical boundaries, the situation can be discussed, even in a more general case, by means of the particular solutions (Eqs. (33)) where the function $f^{(n)}$ is determined by Eq. (28). This equation shows that for $r^2\mu^2 < n(n + 1)$ in the three dimensional case) the functions are nonoscillatory. A similar result applies, of course, for the two dimensional case ($r\mu < m$). Eigensolutions become possible for values of n where the solutions $f^{(n)}$ are in the oscillatory region. In essence, this amounts to

$$r_1^2\mu^2 < n(n+1)$$

For n = 0 one obtains

$$\mu(r_1 - r_0) < \pi/2$$

It follows that the difficulty encountered by the presence of eigensolutions depends in essence on the size of the computed part of the flow field (here $\mu\, r_1$ or $\mu(r_1 - r_0)$), not on the size of the body (here r_0).

Accordingly, it is desirable to keep the computed part of the flow field as small as possible. The limiting factor is the function h(r) in Eq. (54) which must be zero at the location where the farfield condition is applied (except in the case of Klunker's formulation). In realistic cases, the function depends also upon ϕ and this determines the size of the flow field to be computed.

SECTION VII

REMARKS ABOUT THE NUMERICAL SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION

For μ not small it is necessary to solve the flow field by direct methods. Eq. (9) shows that for fixed circular frequency υ, μ becomes very large as the Mach number approaches 1. Here the idea of Achi Brand (Ref. 3) may be useful. One computes the flow field in a mesh which is just fine enough to include all eigenfunctions which are responsible for the failure of an iterative process and uses a finer mesh to derive more accurate solutions by iterations. The iterations will not converge because the contributions of the low eigenvalues are not completely removed and new contributions are brought in by the iteration process itself. For this reason, one alternates between the determination of a correction by means of a direct method in a coarse mesh and an improvement of the solutions by iterations in a fine mesh.

Some thought should be given to the choice of the mesh. The mesh size determines the waves in the solutions which are picked up by the procedure. The mesh size is, therefore, determined by the value of μ. Without further information, this gives a lower limit for the mesh size. This limitation applies to the entire flow field. It can not be disregarded in distant parts of the flow field. The situation is different in steady problems.

The form of the particular solutions (Eq. (33)) shows that the main variation occurs in the r direction. The function $P^m_n(\zeta)\cos(m\theta)$ remains the same along rays through the origin. This means that these particular solutions can be properly represented in a grid system formed by the intersection of equidistant surfaces r = const with rays through the origin.

The same conclusion is obtained by the following consideration. Along the surface the potential has only a finite phase difference mainly determined by the boundary conditions. One can then draw lines of approximately equal phase. They are equidistant curves to the surface. For a plate such lines are shown in Figure 3. Along each of these lines the phase differences are the same. It is therefore not necessary that the number of grid points available along these lines increases as the distance from the plate increases. The number of grid points corresponds to the number of rays through the origin which one would use to approximate the particular solutions Eq. (33) or their two-dimensional counterpart. However, the distance from one such line to the next is critical and cannot be decreased as the distance from the origin increases. Such a system of coordinates would, for instance, be approximated by a system of confocal ellipses and confocal hyperbola as shown in Figure 4. Actually, the ellipses are not equidistant. They must be chosen in such a manner that the largest distance between them is smaller than the critical wave length. Writing down the difference equations for such a system, one obtains a large block tridiagonal matrix. Now one eliminates the unknown values of the potential starting at the surface and proceeding from one ellipse to the next. At the end, one is left with relations between the unknown values of ϕ for the last two ellipses.

In the elimination process one may have some pivoting problems. They would arise, for instance, if for one of the ellipses, the value of μ under consideration is an eigenvalue of the problem with the normal derivative prescribed at the body and zero value of the function prescribed at the particular ellipse. The ultimate aim is to find an expression which represents ϕ and ϕ_n at the outer boundary (or equivalently ϕ along the last two ellipses). This is the information which one obtains from integrating the flow equation with the prescribed boundary conditions at the body. This information is combined with the far field conditions most conveniently in the form 2. This is a global condition which connects all values of ϕ and ϕ_n at the outer boundary. Corresponding to the finite mesh size, one will choose only a limited number of values n and m. From these conditions one then computes the values of ϕ and ϕ_n along the last two ellipses and then by back-substitution for the whole flow field.

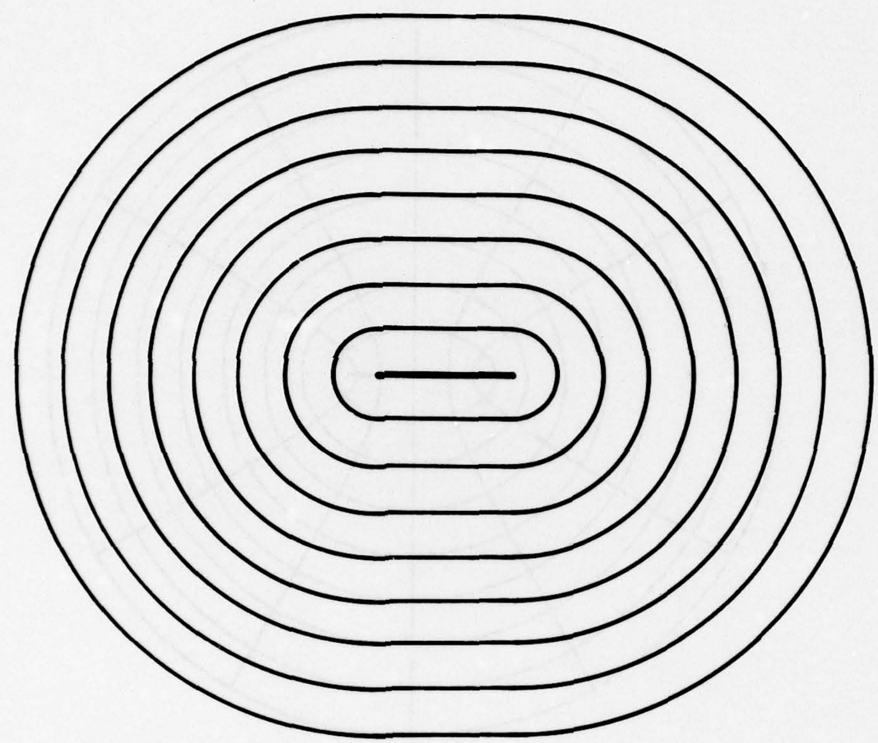


Figure 3. Lines of Nearly Equal Phase for Oscillating Plate in Air at Rest.

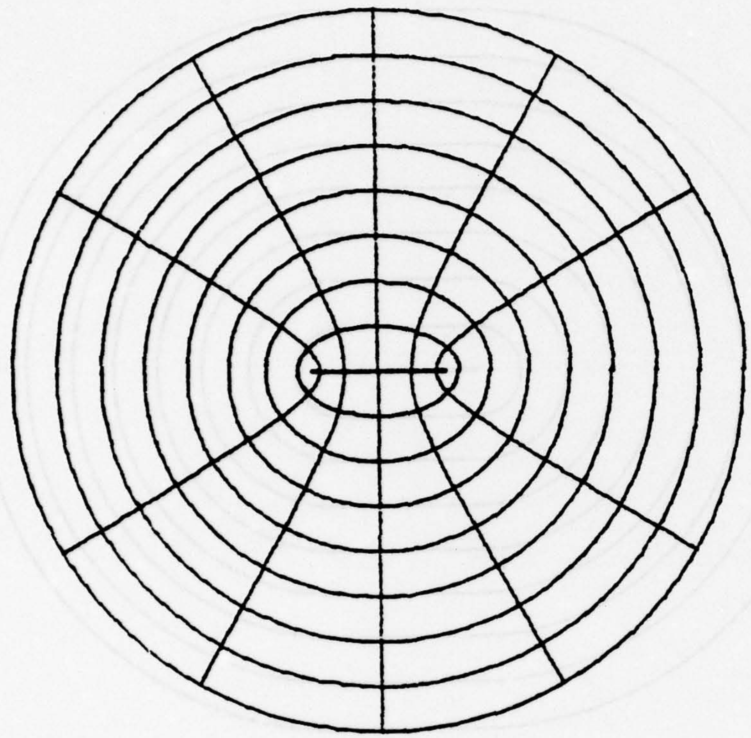


Figure 4. Equi-Focal Ellipses and Hyperbola. (They give a regular system of grid point with some similarity to that of Figure 3.)

It must be conceded that the mesh advocated here is not applicable in all cases. For instance, in the transonic flow problem one may prefer a mesh which takes its orientation from the free stream direction. Moreover, in the transonic flow problem one may have a fairly rapid change of the coefficients of the linearized differential equation for unsteady flows and the mesh must be fine enough to reflect these variations. These questions should be explored by numerical experiments.

SECTION VIII
FINE GRID ITERATIONS

Assume that the problem allows the application of Brand's procedure. Then one will use iterations carried out in a fine mesh to improve approximation obtained in a coarse mesh. It is fortunate that for the fine grid, the far field conditions have approximately a local character although they are, in principle, global conditions. (This means that they give relations between all the values of ϕ and ϕ_n at the outer boundary.)

We use, for this purpose, the third formulation of the far field conditions, letting the surface S coincide with the outer boundary of the flow field. It is assumed that the computation with a coarse mesh has suppressed long wave perturbations which are detrimental to the convergence for iterations in a fine mesh.

In the third formulation, the far field conditions are expressed by means of a density function f(r'), which so far is unknown. Assume that in the k^{th} iteration step we have some approximation $f^{(k)}$ to this density function. Then one makes the following computations:

1. Compute the values of ϕ at the outer pertaining boundary for this function $f^{(h)}$ by means of Eq. (59).

2. Compute the values of ϕ_n at the outer boundary by means of Eq. (59). In this computation a limiting process, familiar from the treatment of solutions of the Laplace equation by means of an integral equation method is required. This function is denoted as $\phi_{n,outer}$.

3. Compute the inner flow field (iteratively) by means of some numerical method using the values of ϕ obtained in step 1 as boundary values.

4. Determine from the solution obtained in step three, the values of ϕ_n at the outer boundary; this constitutes $\phi_{n,inner}$.

5. Compare the result of step 2 with that of step 4. If the difference is sufficiently small, then the far field conditions are satisfied. If this is not the case, then one carries out step 6.

6. Set $f^{(k+1)} = f^{(k)} - (\phi n_{outer} - \phi n_{inner})$

This correction is suggested by the analogy of the Laplace equation. This correction to the pole density gives exactly the jump of the normal derivatives.

The procedure is effective only for wave lengths which are short enough so that the corresponding particular solutions (Eq. (33)) behave in essence like those of the Laplace equation. We illustrate the procedure for the Laplace equation in two dimensions. Let us assume that we have an error in the density function given by

$$\delta f^{(k)} = c \cdot \sin (m\theta)$$

In the manner discussed in Section V, one obtains, as a result of the integration for the far field conditions, a function $\delta\phi$ which is free of singularities inside and outside of the circle with radius r_1, is continuous along the circle, and has a jump of the normal derivative given by the function δf. Such an expression is given by

$$\delta\phi = -\frac{c}{2m} r_1 (r/r_1)^m \sin(m\theta) \quad ; \quad r < r_1$$

$$\delta\phi = -\frac{c}{2m} r_1 (r_1/r)^m \sin(m\theta) \quad ; \quad r > r_1$$

One therefore obtains, as a result of step 1, an error in ϕ

$$\delta\phi = -\frac{c}{2m} r_1 \sin(m\theta)$$

and as a result of step 2, an error in $\phi_{n,\ outer}$

$$\delta\phi_{n,outer} = \frac{c}{2} \sin(m\theta)$$

In step 3, one solves the inner problem with the conditions $\delta\phi_n = 0$ for $r = r_o$, and $\delta\phi$ is given by the result of step 1:

$$\delta\phi = -\frac{c}{2m} r_1 \sin(m\theta) \frac{\left(\frac{r}{r_0}\right)^m + \left(\frac{r_0}{r}\right)^{-m}}{\left(\frac{r_1}{r_0}\right)^m + \left(\frac{r_0}{r_1}\right)^{-m}}$$

This then gives for the normal derivative at $r = r_1$

$$\delta\phi_{m,\,inner} = -\frac{c}{2}\sin(m\theta)\,\frac{\left(\frac{r_1}{r_0}\right)^m - \left(\frac{r_0}{r_1}\right)^m}{\left(\frac{r_1}{r_0}\right)^m + \left(\frac{r_0}{r_1}\right)^m}$$

and for the difference between the normal derivatives of the outer and the inner solution

$$\delta\phi_{m,\,outer} - \delta\phi_{m,\,inner} = \frac{c}{2}\sin(m\theta)\left[1 + \frac{1-\left(\frac{r_0}{r_1}\right)^{2m}}{1+\left(\frac{r_0}{r_1}\right)^{2m}}\right] = c\,\sin(m\theta)\,\frac{1}{1+\left(\frac{r_0}{r_1}\right)^{2m}}$$

The correction carried out in step 6, then leads to an error in the density formula given by

$$\delta f^{(k+1)} = \delta f^{(k)} - c\,\sin(m\theta)\,\frac{1}{1+\left(\frac{r_0}{r_1}\right)^{2m}} = c\,\sin(m\theta)\,\frac{\left(\frac{r_0}{r_1}\right)^{2m}}{1+\left(\frac{r_0}{r_1}\right)^{2m}}$$

This amounts, indeed, to a considerable reduction of the error.

SECTION IX
CONCLUSIONS

The results needed for the practical application are given in Section V. The Klunker-Tracy iteration is useful for small frequencies, but other methods, for instance method 3, may be equally efficient. Most useful is probably a combination of the formulations 2 and 3. The applications are demonstrated in an overly simple example in Section VI. This example may not be very useful for practical work, but it shows possible difficulties, and properly interpreted, it also shows the steps that are to be taken. Section VII tries to counteract the strong analytical flavor of Section VI by discussing in general terms which steps are to be taken in the frame work of a finite difference method. The author believes that the ideas of Achi Brand who advocates the simultaneous use of grid systems of different size can be very useful in practical work. Section VIII describes the use of the third formulation of the far field conditions in an iterative solution for the fine grid.

These discussions are carried out with a view toward practical numerical application. The step to a workable program in a specific situation has not been carried out so far.

APPENDIX

RELATIONS CONNECTING DIFFERENT PARTICULAR SOLUTIONS

The functions $f^{(n)}$ are defined by Eqs. (29), (30), and (32). One has

$$f^{(n)}(r) = r^{-n-1} \exp(-i\mu r) \sum_{k=0} a_k^n r^k \tag{A1}$$

with

$$a_k^n = (2i\mu)^k \frac{1}{k!} \frac{n!}{(n-k)!} \frac{(2n-k)!}{(2n)!} \tag{A2}$$

Notice that for all n

$$a_0^n = 1 ; \quad a_1^n = 2i\mu \tag{A3}$$

The following relations will be derived

$$\frac{df^{(n)}}{dr} = -(n+1) f^{(n+1)} + \frac{\mu^2 n}{4n^2-1} f^{(n-1)} \tag{A4}$$

$$\frac{1}{r} f^{(n)} = f^{(n+1)} + \frac{\mu^2}{4n^2-1} f^{(n-1)} \tag{A5}$$

This is done by substituting the power series development into these formulae. Notice that one has two contributions to $df^{(n)}/dr$; one from differentiating the power series, the other from differentiating the exponential function. One obtains for the coefficient of the term with the power r^{-n-1+k} of df^n/dr

$$(2i\mu)^{k+1} \left\{ \frac{1}{(k+1)!} \frac{n!}{(n-k-1)!} \frac{(2n-k-1)!}{(2n)!} (-n+k) - \frac{1}{2} \frac{1}{k!} \frac{n!}{(n-k)!} \frac{(2n-k)!}{(2n)!} \right\}$$

$$= (2i\mu)^{k+1} \frac{1}{(k+1)!} \frac{n!}{(n-k)!} \frac{(2n-k-1)!}{(2n)!} \left(-n^2 - n + nk - \frac{k^2}{2} + \frac{k}{2}\right)$$

The coefficient of the term with the power r^{-n-1+k} of $df^n/dr + (n+1) f^{n+1}$ is computed next. One obtains

$$(2i\mu)^{k+1}\left\{\frac{1}{(k+1)!}\frac{n!}{(n-k)!}\frac{(2n-k-1)!}{(2n)!}\left(-n^2-n+nk-\frac{k^2}{2}+\frac{k}{2}\right)\right.$$

$$\left.+(n+1)\frac{1}{(k+1)!}\frac{(n+1)!}{(n-k)!}\frac{(2n-k+1)!}{(2n+2)!}\right\}$$

$$=(2i\mu)^{k+1}\frac{1}{(k+1)!}\frac{n!}{(n-k)!}\frac{(2n-k-1)!}{(2n+2)!}\left\{\left(-n^2-n+nk-\frac{k^2}{2}+\frac{k}{2}\right)(2n+1)(2n+2)\right.$$
$$\left.+(n+1)^2(2n-k)(2n-k+1)\right\}$$

The term within the braces is simplified to

$$\{\ \} = n+1\left[\left(-n^2-n+nk-\frac{k^2}{2}+\frac{k}{2}\right)2(2n+1)+(n+1)(2n-k)(2n-k+1)\right]$$

The term within the bracket vanishes for k = -1 and k = 0 because of Eq. (A3). It is a quadratic function in k. Therefore, it can be written as k(k+1) time constant. The constant may depend upon n. It is readily obtained by letting k tend to infinity. One obtains

$$\{\ \} = -n(n+1)k(k+1)$$

The coefficient of the term with power r^{-n-1+k} of $df^n/dr + (n+1)f^{n+1}$ is therefore given by

$$-(2i\mu)^{k+1}\frac{1}{(k-1)!}\frac{n!}{(n-k)!}\frac{(2n-k-1)!}{(2n+1)!}\frac{n}{2}$$

The coefficient of the terms with power r^{-n-1+k} of f^{n-1} comes from a^{n-1}_{k-1}

$$a^{n-1}_{k-1} = (2i\mu)^{k-1}\frac{1}{(k-1)!}\frac{(n-1)!}{(n-k)!}\frac{(2n-k-1)!}{(2n-2)!}$$

This suffices to establish Eq. (A4). Eq. (A5) is verified in an analogous manner. The coefficient of the term with power r^{-n-2+k} of $f^{n+1} - (1/r)f^n$ is given by

$$a^{n+1}_k - a^n_k = (2i\mu)^k\frac{1}{k!}\left\{\frac{(n+1)!}{(n-k+1)!}\frac{(2n-k+2)!}{(2n+2)!}-\frac{n!}{(n-k)!}\frac{(2n-k)!}{(2n)!}\right\}$$

$$=(2i\mu)^k\frac{1}{k!}\frac{n!\,(2n-k)!}{(n-k+1)!\,(2n+2)!}(n+1)\{(2n-k+1)(2n-k+2)-2(2n+1)(n-k+1)\}$$

The term in braces vanishes for $k = 0$ and $k = 1$ and behaves as k^2 for $k \to \infty$. Hence,

$$\{\} = k(k-1)$$

and

$$a_k^{n+1} - a_k^n = (2i\mu)^k \frac{1}{(k-2)!} \frac{n!}{(n-k+1)!} \frac{(2n-k)!}{(2n+1)!} \frac{1}{2}$$

The coefficient with the power r^{-n-2+k} of f^{n-1} is given by a_{k-2}^{n-1}

$$a_{k-2}^{n-1} = (2i\mu)^{k-2} \frac{1}{(k-2)!} \frac{(n-1)!}{(n-k+1)!} \frac{(2n-k)!}{(2n-2)!}$$

From the last two formulae, Eq. (A5) is immediately obtained. From a theoretical point of view the existence of formulae of the type A4 and A5 is not surprising. One deals with contiguous confluent hypergeometric functions.

Similar relations exist for the Legendre functions. We use the definitions given in Reference 4, but take the formulae from Reference 5. There exist a number of different definitions in the literature (also, there is probably a misprint in the section called notations in Chapter 8 of Reference 5). For clarity, the definitions to be used and the basic formulae are repeated here.

$$P_n(\zeta) = \sum_{r=0} \frac{(-)^r (2n-2r)!}{2^n r! (n-r)! (n-2r)!} \tag{A6}$$

Reference 4, page 274. We have not given an upper limit for r. The sum terminates automatically when one of the factorials in the denominator assumes a negative argument (here for $r = \frac{n}{2} + 1$ if n is even, and for $r = \frac{n+1}{2}$ if n is odd). That is, the sum goes up to $r = \frac{n}{2}$ for even n, and up to $r = \frac{n-1}{2}$ for odd n. Furthermore,

$$P_n^m = (-)^m (1-\zeta^2)^{m/2} \frac{d^m P_n(\zeta)}{d\zeta^m} \tag{A7}$$

Differences in definition may occur because one may replace $(-1)^m$ by 1, and because of the argument $(1-\zeta^2)^{m/2}$ (in the formulae of

Reference 4 one has always (ζ^2-1)). This would introduce a factor $(\pm i)^m$. For this reason, the formulae of Reference 5, made consistent with the present definitions, are repeated here. The signs have been checked by evaluating these expressions for the highest powers of ζ. Also, the lowest powers have been examined.

$$P_n^{m+1}(\zeta) = (1-\zeta^2)^{-1/2}\{(n-m)\zeta P_n^m(\zeta) - (n+m)P_{n-1}^m(\zeta)\} \tag{A8}$$

from (8.5.1) of Reference 5

$$(1-\zeta)^2 \frac{dP_n^m}{d\zeta} - m\zeta P_n^m = (n+m)(n-m+1)(1-\zeta^2)^{1/2} P_n^{m-1} \tag{A9}$$

from (8.5.2) of Reference 5

$$\zeta P_n^m = \frac{1}{2n+1}\{(n-m+1)P_{n+1}^m + (n+m)P_{n-1}^m\} \tag{A10}$$

from (8.5.3) of Reference 5

(no check necessary since the superscript of P_n^m does not change)

$$(1-\zeta^2)\frac{dP_n^m}{d\zeta} = -n\zeta P_n^m + (n+m)P_{n-1}^m \tag{A11}$$

from (8.5.4) of Reference 5

(no check necessary, since the superscript of P_n^m does not change)

$$P_{n+1}^m - P_{n-1}^m = -(2n+1)(1-\zeta^2)^{1/2} P_n^{m-1} \tag{A12}$$

from (8.5.5) of Reference 5

The following formulae will be used.

$$\zeta P_n^m = \frac{1}{2n+1}[(n-m+1)P_{n+1}^m + (n+m)P_{n-1}^m] \tag{A13}$$

$$(1-\zeta^2)\frac{dP_n^m}{d\zeta} = \frac{1}{2n+1}[-n(n-m+1)P_{n+1}^m + (n+1)(n+m)P_{n-1}^m] \tag{A14}$$

$$(1-\zeta^2)^{1/2}P_n^m = \frac{1}{2n+1}[P_{n+1}^{m+1} + P_{n-1}^{m+1}] \tag{A15}$$

$$(1-\zeta^2)^{3/2}\zeta\frac{dP_n^m}{d\zeta}=\frac{1}{2n+1}\left[-(n(1-\zeta^2)-m)P_{n+1}^{m+1}-((n+1)(1-\zeta^2)+m)P_{n-1}^{m+1}\right] \tag{A16}$$

$$(1-\zeta^2)^{1/2}P_n^m=\frac{1}{2n+1}\left[(n-m+1)(n-m+2)P_{n+1}^{m-1}-(n+m-1)(n+m)P_{n-1}^{m-1}\right] \tag{A17}$$

$$\zeta(1-\zeta^2)^{3/2}\frac{dP_n^m}{d\zeta}=\frac{1}{2n+1}\Big[(n-m+1)(n-m+2)[n(1-\zeta^2)+m]P_{n+1}^{m-1}$$
$$+(n+m-1)(n+m)[(n+1)(1-\zeta^2)-m]P_{n-1}^{m-1}\Big] \tag{A18}$$

Eq. (A13) is identical with Eq. (A10). Eq. (A14) follows from Eq. (A11) with some further computations by eliminating $\zeta P_n^m(\zeta)$ by means of Eq. (A10). Eq. (A15) is identical with Eq. (A12), with m replaced by m + 1. Eq. (A16) requires some further arimetic. One obtains from Eq. (A7)

$$\frac{dP_n^m}{d\zeta}=\frac{-m\zeta}{1-\zeta^2}P_n^m-\frac{1}{(1-\zeta^2)^{1/2}}P_n^{m+1}$$

Hence,

$$\zeta(1-\zeta^2)^{3/2}\frac{dP_n^m}{d\zeta}+m\zeta^2(1-\zeta^2)^{1/2}P_n^m+(1-\zeta^2)P_n^{m+1}$$

Now from Eq. (A13) with m replaced by m + 1

$$\zeta P_n^{m+1}=\frac{1}{2n+1}\left[(n-m)P_{n+1}^{m+1}+(n+m+1)P_{n-1}^{m+1}\right]$$

Furthermore, we use Eq. (A15). Substituting these two expressions into the last equation, we ultimately obtain Eq. (A16). To obtain Eq. (A17) one first writes down Eq. (A8) with m replaced by m-1

$$(1-\zeta^2)^{1/2}P_n^m=\left[(n-m+1)\zeta P_n^{m-1}(\zeta)-(n+m-1)P_{m-1}^{m-1}(\zeta)\right]$$

and substitutes Eq. (A10) with m replaced by m-1

$$\zeta P_n^{m-1}=\frac{1}{2n+1}\left[(n-m+2)P_{n+1}^{m-1}+(n+m-1)P_{m-1}^{m-1}\right]$$

To derive Eq. (A18) we start from Eq. (A9)

$$(1-\zeta^2)\frac{dP_n^m}{d\zeta} = m\zeta P_n^m + (n+m)(n-m+1)(1-\zeta^2)^{1/2} P_n^{m-1}$$

Now from Eq. (A8) with m replaced by m-1

$$P_n^m = (1-\zeta^2)^{-1/2}[(n-m+1)\zeta P_n^{m-1} - (n+m-1)P_{n-1}^{m-1}]$$

Hence

$$(1-\zeta^2)^{3/2}\frac{dP_n^m}{d\zeta} = [m(n-m+1)\zeta^2 P_n^{m-1} - m(n+m-1)\zeta P_{n-1}^{m-1}$$

$$+ (n+m)(n-m+1)(1-\zeta^2)P_n^{m-1}]$$

$$= (1-\zeta^2)(n-m+1)nP_n^{m-1} + m(n-m+1)P_n^{m-1} - m(n+m-1)\zeta P_{n-1}^{m-1}]$$

Now from Eq. (A10) with m replaced by m-1

$$\zeta P_n^{m-1} = \frac{1}{2n+1}[(n-m+2)P_{n+1}^{m-1} + (n+m-1)P_{n-1}^{m-1}]$$

This leads to Eq. (A18).

These equations are now used to derive formulae which express x, y, or z derivatives of certain particular solutions by other particular solutions with changed m or n. We have

$$\zeta = \frac{z}{r}, \quad r^2 = x^2+y^2+z^2, \quad \operatorname{arctg}\psi = \frac{y}{x}$$

$$x = (x^2+y^2)^{1/2}\cos\psi \tag{A19}$$

$$y = (x^2+y^2)^{1/2}\sin\psi$$

Hence

$$\frac{\partial r}{\partial x} = \frac{x}{r} = (1-\zeta^2)^{1/2}\cos\psi$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = (1-\zeta^2)^{1/2}\sin\psi$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \zeta \tag{A20}$$

$$\frac{\partial \xi}{\partial x} = \frac{-z}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r}\xi(1-\xi^2)^{1/2}\cos\psi$$

$$\frac{\partial \xi}{\partial y} = \frac{-z}{r^2}\frac{\partial r}{\partial y} = -\frac{1}{r}\xi(1-\xi^2)^{1/2}\sin\psi$$

$$\frac{\partial \xi}{\partial z} = \frac{1}{r} - \frac{z}{r^2}\frac{\partial r}{\partial z} = \frac{1}{r}(1-\xi^2) \tag{A20}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{1}{r}(1-\xi^2)^{-1/2}\sin\psi \qquad \text{(concluded)}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{r}(1-\xi^2)^{-1/2}\cos\psi$$

$$\frac{\partial \theta}{\partial z} = 0$$

Using the formulae just derived, one obtains

$$\frac{\partial}{\partial z}\left[P_n^m(\xi)\left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\} f^{(n)}(r)\right] = \left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\}\left[\frac{dP_n^m}{d\xi}\frac{\partial \xi}{\partial z} f^{(n)}(r) + P_n^m(\xi)\frac{df^{(n)}}{dr}\frac{\partial r}{\partial z}\right]$$

$$= \left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\}\left[(1-\xi^2)\frac{dP_n^m}{d\xi}\frac{f^{(n)}(r)}{r} + \xi P_n^m(\xi)\frac{df^{(n)}}{dr}\right]$$

Here Eqs. (A4), (A5), (A13) and (A14) are substituted.

$$\frac{\partial}{\partial z}\left[P_n^m(\xi)\left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\} f^{(n)}(r)\right] =$$

$$\left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\}\left\{\left[-\frac{n(n-m+1)}{2n+1}P_{n+1}^m + \frac{(n+1)(n+m)}{2n+1}P_{n-1}^m\right]\left[f^{n+1} + \frac{\mu^2}{4n^2-1}f^{n-1}\right]\right.$$

$$\left. + \left[\frac{n-m+1}{2n+1}P_{n+1}^m + \frac{n+m}{2n+1}P_{n-1}^m\right]\left[-(n+1)f^{n+1} + \frac{\mu^2 n}{4n^2-1}f^{n-1}\right]\right\}$$

Here the products $P_{n+1}^m \; f^{n-1}$ and $P_{n-1}^m \; f^{n+1}$ vanish and one finds

$$\frac{\partial}{\partial z}\left[P_n^m(\xi)\left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\} f^{(n)}(r)\right] = \left[-(n-m+1)P_{n+1}^m(\xi)f^{n+1}(r) + \right.$$

$$\left. + \frac{\mu^2}{4n^2-1}(n+m)P_{n-1}^m(\xi)f^{(n-1)}(r)\right]\left\{\begin{matrix}\cos m\psi\\ \sin m\psi\end{matrix}\right\} \tag{A21}$$

One observes that $P_n^m = 0$ if $n < m$; that is, this sequence of equations starts with $m = n$ and then expresses higher derivatives in terms of function P_n^m with higher values of n.

Next consider

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos m\psi\right] = \left[\frac{dP_n^m}{d\zeta}(-\zeta)(1-\zeta^2)^{1/2}\frac{f^{(n)}(r)}{r} + P_n^m(\zeta)\frac{df^{(n)}}{dr}(1-\zeta^2)^{1/2}\right]\cos(m\psi)\cos\psi$$
$$+ m P_n^m(\zeta)\frac{f^{(n)}(r)}{r}(1-\zeta^2)^{-1/2}\sin(m\psi)\sin\psi$$

$$\frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin m\psi\right] = \left[\frac{dP_n^m}{d\zeta}(-\zeta)(1-\zeta^2)^{1/2}\frac{f^{(n)}(r)}{r} + P_n^m(\zeta)\frac{df^{(n)}}{dr}(1-\zeta^2)^{1/2}\right]\sin(m\psi)\sin\psi$$
$$+ m P_n^m(\zeta)\frac{f^{(n)}(r)}{r}(1-\zeta^2)^{-1/2}\cos(m\psi)\cos\psi$$

Hence,

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos(m\psi)\right] - \frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin(m\psi)\right]$$
$$= \left\{\left[\frac{dP_n^m}{d\zeta}(-\zeta)(1-\zeta^2)^{1/2} - m P_n^m(\zeta)(1-\zeta^2)^{-1/2}\right] f^{(n)}(r) + P_n^m(\zeta)(1-\zeta^2)^{1/2}\frac{\partial f^{(n)}}{dr}\right\}\cos((m+1)\psi) \tag{A22}$$

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos(m\psi)\right] + \frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin(m\psi)\right]$$
$$= \left\{\left[\frac{dP_n^m}{d\zeta}(-\zeta)(1-\zeta^2)^{1/2} + m P_n^m(\zeta)(1-\zeta^2)^{-1/2}\right] f^{(n)}(r) + P_n^m(\zeta)(1-\zeta^2)^{1/2}\frac{df^{(n)}}{dr}\right\}\cos((m-1)\psi) \tag{A23}$$

Now we substitute into Eq. (A22) the expressions Eqs. (A15) and (A16) and (A4) and (A5).

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos(m\psi)\right] - \frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin(m\psi)\right] =$$

$$= \frac{\cos((m+1)\psi)}{2n+1}\Big\{\Big[\frac{1}{1-\zeta^2}\big[n(1-\zeta^2)-m\big] P_{n+1}^{m+1} + \big((n+1)(1-\zeta^2)+m\big)P_{n-1}^{m+1}\Big]$$

$$+ \frac{m}{1-\zeta^2}\left(P_{n+1}^{m+1} - P_{n-1}^{m+1}\right)\Big]\Big[f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1} f^{(n-1)}(r)\Big]$$

$$+ \left[P_{n+1}^{m+1} - P_{n-1}^{m+1}\right]\Big[(n+1)f^{(n+1)}(r) - \frac{\mu^2 n}{4n^2-1} f^{(n-1)}(r)\Big]\Big\}$$

$$= \frac{\cos((m+1)\psi)}{2n+1}\Big\{\left[n P_{n+1}^{m+1} + (n+1)P_{n-1}^{m+1}\right]\Big[f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1} f^{(n-1)}(r)\Big]$$

$$+ \left[P_{n+1}^{m+1} - P_{n-1}^{m+1}\right]\Big[(n+1)f^{(n+1)}(r) - \frac{\mu^2 n}{4n^2-1} f^{(n-1)}(r)\Big]\Big\}$$

$$= \cos((m+1)\psi)\left[P_{n+1}^{m+1} f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1} P_{n-1}^{m+1} f^{(n-1)}(r)\right]$$

Hence

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos(m\psi)\right] - \frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin(m\psi)\right] \tag{A24}$$

$$= \left[P_{n+1}^{m+1}(\zeta) f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1} P_{n-1}^{m+1}(\zeta) f^{(n-1)}(r)\right]\cos((m+1)\psi)$$

Take the expression (A23) and substitute Eqs. (A17) and (A18)

$$\frac{\partial}{\partial x}\left[P_n^m(\zeta) f^{(n)}(r)\cos(m\psi)\right] + \frac{\partial}{\partial y}\left[P_n^m(\zeta) f^{(n)}(r)\sin(m\psi)\right]$$

$$= \frac{\cos((m-1)\psi)}{2n+1}\Big\{-\frac{1}{1-\zeta^2}\Big[(n-m+1)(n-m+2)\left[n(1-\zeta^2)+m\right]P_{n+1}^{m-1}$$

$$+ (n+m-1)(n+m)\left[(n+1)(1-\zeta^2)-m\right]P_{n-1}^{m-1}\Big]$$

$$+ \frac{m}{1-\zeta^2}\Big[(n-m+1)(n-m+2)\,P_{n+1}^{m-1}$$

$$- (n+m-1)(n+m)\,P_{n-1}^{m-1}\Big]\Big[f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1} f^{(n-1)}(r)\Big]$$

$$+ \left[(n-m+1)(n-m+2)P_{n+1}^{m-1} - (n+m-1)(n+m)P_{n-1}^{m-1}\right]\times$$

$$\Big[-(n+1)f^{(n+1)}(r) + \frac{\mu^2 n}{4n^2-1} f^{(n-1)}(r)\Big]\Big\}$$

$$= \frac{\cos((m-1)\psi)}{2n+1}\{[-(n-m+1)(n-m+2)n\,P^{m-1}_{n+1} - (n+m-1)(n+m)(n+1)P^{m-1}_{n-1}]\times$$

$$[f^{(n+1)}(r) + \frac{\mu^2}{4n^2-1}f^{n-1}(r)]$$

$$+[(n-m+1)(n-m+2)P^{m-1}_{n+1} - (n+m-1)(n+m)P^{m-1}_{n-1}]\times$$

$$[-(n+1)f^{(n+1)}(r) + \frac{\mu^2 n}{4n^2-1}f^{(n-1)}(r)]\}$$

$$= \cos((m-1)\psi)[-(n-m+1)(n-m+2)P^{m-1}_{n+1}f^{(n+1)} - (n+m-1)(n+m)P^{m-1}_{n-1}f^{(n-1)}$$

Hence

$$\frac{\partial}{\partial x}[P^m_n(\xi)f^{(n)}(r)\cos(m\psi)] + \frac{\partial}{\partial y}[P^m_n(\xi)f^{(n)}(r)\sin(m\psi)]$$

$$= [-(n-m+1)(n-m+2)P^{m-1}_{n+1}(\xi)f^{(n+1)}(r) - (n+m-1)(n+m)P^{m-1}_{n-1}(\xi)f^{(n-1)}(r)]\times$$

$$\times\cos((m-1)\psi) \tag{A25}$$

There exist, of course, analogous formulae with sine and cosine interchanged (and certain changes of signs).

Similar, but much simpler relations exist for the two-dimensional case. One has

$$r^2 = x^2 + y^2 \quad , \quad \theta = \mathrm{arctg}(y/x)$$

$$\frac{\partial r}{\partial x} = \cos\theta \quad , \quad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta \, , \quad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta$$

Then one finds

$$\frac{\partial}{\partial x}[H^{(2)}_m(\mu r)\cos(m\theta)] = \mu\frac{dH^{(2)}_m(\mu r)}{d(\mu r)}\cos(m\theta)\cos\theta + \frac{m}{r}H^{(2)}_m(\mu r)\sin(m\theta)\sin\theta$$

$$\frac{\partial}{\partial y}[H^{(2)}_m(\mu r)\sin(m\theta)] = \mu\frac{dH^{(2)}_m(\mu r)}{d(\mu r)}\sin(m\theta)\sin\theta + \frac{m}{r}H^{(2)}_m(\mu r)\cos(m\theta)\cos\theta$$

Hence

$$\frac{\partial}{\partial x}\left[H_m^{(2)}(\mu r)\cos(m\theta)\right] - \frac{\partial}{\partial y}\left[H_m^{(2)}(\mu r)\sin(m\theta)\right]$$

$$= \left[\mu \frac{dH_m^{(2)}(\mu r)}{d(\mu r)} - \frac{m}{r}H_m^{(2)}(\mu r)\right]\cos[(m+1)\theta] \tag{A26}$$

$$\frac{\partial}{\partial x}\left[H_m^{(2)}(\mu r)\cos(m\theta)\right] + \frac{\partial}{\partial y}\left[H_m^{(2)}(\mu r)\sin(m\theta)\right]$$

$$= \left[\mu \frac{dH_m^{(2)}(\mu r)}{d(\mu r)} + \frac{m}{r}H_m^{(2)}(\mu r)\right]\cos[(m-1)\theta] \tag{A27}$$

Consider some linear combination

$$Z_p(\alpha x) = c_1 J_p(\alpha x) + c_2 N_p(\alpha x)$$

(c_1 and c_2 constant real or complex coefficients, J_p and N_p are respectively Bessel and Neumann functions of order p). One has the following relations

$$\frac{d}{dx}(x^p Z_p(\alpha x)) = \alpha x^p Z_{p-1}(\alpha x) \qquad \frac{p}{x}Z_p(\alpha x) + \alpha\frac{dZ_p(\alpha x)}{d(\alpha x)} = \alpha Z_{p-1}(\alpha x)$$

$$\frac{d}{dx}(x^{-p}Z_p(\alpha x)) = -\alpha x^{-p}Z_{p+1}(\alpha x) \qquad -\frac{p}{x}Z_p(\alpha x) + \alpha\frac{dZ_p(\alpha x)}{d(\alpha x)} = -\alpha Z_{p+1}(\alpha x)$$

$H_m^{(2)}$ is such a linear combination of J_m and N_m. One therefore obtains from Eqs. (A26) and (A27)

$$\frac{\partial}{\partial x}\left[H_m^{(2)}(\mu r)\cos(m\theta)\right] - \frac{\partial}{\partial y}\left[H^{(2)}(\mu r)\sin(m\theta)\right] = -\mu H_{m+1}^{(2)}(\mu r)\cos[(m+1)\theta] \tag{A28}$$

$$\frac{\partial}{\partial x}\left[H_m^{(2)}(\mu r)\cos(m\theta)\right] + \frac{\partial}{\partial y}\left[H^{(2)}(\mu r)\sin(m\theta)\right] = \mu H_{m+1}^{(2)}(\mu r)\cos[(m-1)\theta] \tag{A29}$$

There exist, of course, analogous formulae with sine and cosine interchanged (and some changes in sign).

REFERENCES

1. Klunker, E. B., Contributions to Methods for Calculating the Flow About Thin Lifting Wings at Transonic Speeds - Analytic Expressions for the Far Field, NASA TN D-6530, Nov. 1971.

2. Traci, R. M., E. D. Albano, J. L. Farr Jr., and H. Y. Cheng, Small Distrubance Transonic Flows About Oscillating Airfoils, AFFDL-TR-74,37, Air Force Flight Dynamics Laboratory, Air Force Systems Command, Wright Patterson Air Force Base, Ohio 45433.

3. Brand, Achi, "Multilevel Adaptive Solutions to Boundary Value Problems," Mathematics of Computation, Vol. 31, No. 138, April 1977, pp. 333-390.

4. Copson, E. T., An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, 1935, Chapter XI.

5. Abramowith, Milton, and Stegun, Irene A., editors, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, National Bureau of Standards Applied Mathematics Series 55, 1964.

☆U.S.Government Printing Office: 1979 — 657-084/314